View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Unisa Institutional Repository

LINEAR PERTURBATIONS OF A SCHWARZSCHILD BLACK HOLE

by

AMOS SOWETO KUBEKA

submitted in fulfillment of the requirements for
the degree of

MASTER OF SCIENCE

in the subject

APPLIED MATHEMATICS

at the

UNIVERSITY OF SOUTH AFRICA

SUPERVISOR: PROF N.T. BISHOP
JOINT SUPERVISOR: PROF W.M. LESAME

OCTOBER 2006

Abstract

We firstly numerically recalculate the Ricci tensor of non-stationary axisymmetric space-times (originally calculated by Chandrasekhar) and we find some discrepancies both in the linear and non-linear terms. However, these discrepancies do not affect the results concerning linear perturbations of a Schwarzschild black hole. Secondly, we use these Ricci tensors to derive the Zerilli and Regge-Wheeler equations and use the Newman-Penrose formalism to derive the Bardeen-Press equation. We show the relation between these equations because they describe the same linear perturbations of a Schwarzschild black hole. Thirdly, we illustrate heuristically (when the angular momentum (l) is 2) the relation between the linearized solution of the Einstein vacuum equations obtained from the Bondi-Sachs metric and the Zerilli equation, because they describe the same linear perturbations of a Schwarzschild black hole. Lastly, by means of a coordinate transformation, we extend Chandrasekhar's results on linear perturbations of a Schwarzschild black hole to the Bondi-Sachs framework.

Key terms:

Bondi-Sachs metric; Schwarzschild black hole; Einstein vacuum equations, linear solution; Black hole theory; black hole perturbation theory; gravitational radiation; linear perturbations.

Acknowledgement

I would like to thank

- GOD for everything.
- My parents for the moral support.
- My supervisor Prof. N. T Bishop and co-supervisor Prof W. M Lesame for their encouragement, valuable discussions, criticisms and support. I find it always difficult to express my gratitude to someone who is intimately involved in my intellectual development. In this instance the gratitude is both sincere and great.
- NRF and UNISA for the financial support.

Lastly, I would like to dedicate this dissertation to my family Zandile, Nonhlahla, Themba, Jabulane, Mphonyana, Phillip for their love and support in times of need.

Contents

List of Figures

List of Tables

Preface

There has been much progress in black hole research from the early 1960s until the present. A great deal has been learnt since then, from the global structure of equilibrium black holes to linear perturbation theory and black-hole thermodynamics to new black hole models incorporating matter through scalar fields. As people came to understand better the physical concept of a black hole in the 1950's, they started studying space-times that represented perturbations of the geometry of a black hole. Regge and Wheeler [70] were the first to study these perturbations, and were able to find a formulation for one-half of the degrees of freedom of the problem. The formulation for the other half had to wait until the work of Zerilli [100]. Perturbations of flat space-time had been considered earlier [54], and in fact Einstein himself discovered an approximate version of the Schwarzschild solution as a perturbation of flat space.

The motivation of Regge and Wheeler to study perturbations of black holes was to assess the stability of the black hole solutions. The studies of Regge, Wheeler and Zerilli, and later on of Price [68] and others indeed showed that black hole solutions are stable under small perturbations. Also, stability of black hole solutions was the motivation of the early work on second-order perturbation analysis of the Schwarzschild solution with the aim of probing the non-linear stability of the black hole horizon.

The perturbations of the "exterior" Schwarzschild metric could be coupled to perturbations of interior solutions, and therefore one could analyze perturbations of stellar objects. This is a subject that lies outside the scope of this dissertation. For further references see the work of Cunningham [26] and for more recent references see Tominaga [87]. Also outside the scope of this dissertation is the study of perturbations of rotating black holes, originally studied by Teukolsky [85] in 1973, and recently extended to second order perturbations by Campanelli *et al.* [18] in 1999.

The perturbations of a Schwarzschild black hole space-time can be described by two "potential-form" equations called Zerilli and Regge-Wheeler equations [24, 25, 99, 100]. These equations differ only in the details of their potentials. For any given value of the angular momentum l there are two independent classes of perturbation characterised by their parities $(-1)^l$ (even parity) and $(-1)^{l+1}$ (odd parity) [89]. The Zerilli equation arose initially in the study of even-parity perturbations of the Schwarzschild solution [59], while the Regge-Wheeler equation arose in the study of odd-parity perturbations in the same formalism. A very different approach is based on the Newman-Penrose projections on null tetrads. This formalism leads to the Bardeen-Press equation [7, 24, 25], which is an equation of a different form to that of Zerilli or Regge-Wheeler. The Zerilli, Regge-Wheeler and Bardeen-Press equations, all describe the same linear perturbations of a Schwarzschild black hole, and Chandrasekhar [24, 25] showed explicitly the connection between the Zerilli and Bardeen-Press equations, and between the Regge-Wheeler and Bardeen-Press equations, and between the Regge-Wheeler and Zerilli equations. Many of the original papers in this area did not make clear these connections, however, an excellent presentation which includes an exhaustive treatment of the results and connections among these equations appears in the monograph by Chandrasekhar [25].

The existence of different descriptions of perturbations of black holes led Chandrasekhar [23] (see also [25], sec. 28) to consider the general question of the relationship between the two potentials: the Zerilli potential and Regge-Wheeler potential which are "equivalent" in the sense that they produce the same physical consequence (more specifically, they have the same reflection and transmission coefficients). He found that a sufficient condition for the Zerilli potential and Regge-Wheeler potential to be equivalent is that they are related by

$$V^{(\pm)} = \pm\beta \tfrac{df}{dr_*} + \beta^2 f^2 + \kappa f,$$

where (+) refers to the Zerilli potential and (-) to the Regge-Wheeler potential, f is some function of r_*, and β, κ are constants. The above relationship is identical to the relationship of two potential-form equations in Anderson's formalism [1] of "intertwining operators". This formalism, however, gives a slightly different viewpoint on the issue in the sense that it clarifies the fact that for any potential, there are equivalent potentials, in fact, an infinite number of equivalent potentials. For Schwarzschild black holes, the Zerilli and Regge-Wheeler potentials are only two of an infinite set of possible potentials [1].

This range of possibilities raises a question that is not only of interest as a matter of principle, but which has important practical consequences. With laser interferometric gravity-wave detection renewed attention has been given to quasi-normal modes of black holes since quasi-normal oscillations may provide a signal with a signature characterizing their black hole origin (see Edelstein and Vishveshwara [32]).

The computation of quasi-normal frequencies, and other aspects of quasi-normal excitations, can involve numerical problems. Analytic or semi-analytic approaches are often far superior to brute force numerical computations. One clear example of this is the high-precision computation of the Schwarzschild quasi-normal frequencies by Leaver [55]. Leaver's approach is to seek a solution of the Regge-Wheeler equation in the form of a power series entirely in terms of the radius r. After certain simplifications (in particular, after the behavior at infinity is factored out) the equation leads to a three-term recursion relation for the coefficients in the power-series solution. Three-term recursion relations can be treated by continued-fraction methods. With such a method, Leaver shows that the quasi-normal frequencies are those complex values of σ for which the continued fractions converge. He then uses the convergence of the continued fraction as the basis for a precise and stable computation scheme for the quasi-normal frequencies. Chandrasekhar [24, 25] used numerical integration methods to compute these quasi-normal modes frequencies, and the theory behind this method shall be discussed later in detail in chapter 3. chapter 3 is based entirely on this literature review.

In studying linear perturbations of a Schwarzschild black hole we are able to study its static space-time properties and the emission of gravitation radiation. The gravitational radiation emitted by a Schwarzschild black hole carries information about its mass (as well as spin and charge for rotating and/or charged black holes). Also by studying the perturbations of a Schwarzschild black hole it is possible to make conclusions about the stability of the Einstein equations [82]. Because of the challenges of studying the gravitational radiation analytically, people have developed numerical relativity [83] (which is concerned with the numerical simulation of Einstein field equations) to solve these problems. In this dissertation we study linear perturbations of a Schwarzschild black hole that produce gravitational radiation. We do that by firstly studying the space-time structure of a Schwarzschild black hole to get a full geometrical understanding of the space-time. Then secondly we derive Zerilli, Regge-Wheeler, and Bardeen-Press equations and show that they are related since they describe the same physical phenomena (Schwarzschild linear perturbations). In the process we use Maple to recalculate Ricci tensors of axisymmetric space-times which were analytically calculated by Chandrasekhar [25] in 1972. The results have been included in the Appendices A and B. Lastly, we transform both even and odd parity perturbations of a Schwarzschild black hole to the Bondi-Sachs frame-work and in the process we get Bondi-Sachs metric variables that describe the gravitational radiation information.

The layout of this dissertation is as follows: In chapter 1 we introduce curvature, the Weyl tensor, Einstein vacuum field equations, the use of computer algebraic system (Maple) in solving Einstein field equations, the Newman-Penrose formalism, and an overview of the Bondi-Sachs metric. These concepts shall form the foundation for this dissertation.

In chapter 2, we introduce spherically symmetric space-times, the Schwarzschild solution, Birkhoff's theorem, Schwarzschild geometry, Carter-Penrose diagrams and causal properties, and lastly we describe Schwarzschild geometry in the Newman-Penrose formalism with the aim of deriving the Bardeen-Press equation in chapter 3.

In chapter 3, we introduce Ricci and Einstein tensors for non-stationary axisymmetric space-times. We then derive Regge-Wheeler and Zerilli equations independently in the same formalism, and we show the relation between these two equations.

Also we derive the Bardeen-Press equation using a different formalism (Newman-Penrose formalism) and thereafter we derive it from the Zerilli equation, and by so doing we show the relation between them even though they were derived using different formalisms. We also show the relation between the Bardeen-Press and Regge-Wheeler equations by deriving the Bardeen-Press equation from the Regge-Wheeler equation. Lastly we introduce quasi-normal modes of a Schwarzschild black hole.

In chapter 4, we introduce the Bondi-Sachs metric and linearized vacuum field equations for the Bondi-Sachs metric and we solve them to obtain a linearized solution which is important for chapter 5.

In chapter 5, we illustrate heuristically the relation between the linearized solution obtained in chapter 4 and the Zerilli equation since they describe the same Schwarzschild linear perturbations. We then transform even and odd-parity perturbations derived in chapter 3 to Bondi-Sachs form. Finally, we discuss the results of this chapter.

In chapter 6, we summarize the whole dissertation and we discuss the implications of the work of this dissertation for similar projects in the future.

The notational conventions in this dissertation will be as follows: for the upper case Latin letters (A, B) the range is (2,3); for the lower case Latin indices $(a, b, c, etc.)$ and (i, j, k, etc), the ranges are (1,2,3,4) and (0,1,2,3) respectively.

Table 1: Symbols used (with the page number on which the symbol is first used)

Symbol	Representation	Notation Page
$\mathcal{M}$	four-dimensional differential manifold	2
S^2	2-sphere manifold with standard metric g_S	17
N	2-dimensional manifold with Lorentzian metric g_N	17
$\widetilde{\mathcal{M}}$	conformal 'compactification' of $\mathcal{M}$	27
$\mathcal{Y}$	curve on $\mathcal{M}$	3
$\mathbf{A}^i$,$\mathbf{U}^i$	four vectors	2
$\mathbf{K}_k$,$\mathbf{V}^k$	four vectors	4,5
E	scalar	5
R	Ricci scalar	6
Λ	real scalar	10
Φ_{ij}	complex-scalar	10
$\Psi_0 ... \Psi_4$	five complex scalars	11
Γ^k_{ij}	connection (Christoffel symbols)	4
g_{ij}	Lorentzian metric tensor	2
$\widetilde{g}$	conformal 'compactification' of g	27
$R^l{}_{ijk}$	Riemann tensor	4
R_{ij}	Ricci tensor	6
$\mathbf{F}_{ij}$	electromagnetic field tensor	3
T^l_{ij}	torsion tensor	4
G_{ij}	Einstein tensor	7
$Z^{(+)}$	1-dimensional Schrodinger wave function (for Zerilli equation)	40
$Z^{(-)}$	1-dimensional Schrodinger wave function (for Regge-Wheeler equation)	37
$\cdot$	differentiation with respect to the parameter s of the curve $\mathcal{Y}$	3

Symbol	Representation	Notation Page
$'$	differentiation with respect to r	20
$V^{(+)}$	potential for Zerilli equation	41
$V^{(-)}$	potential for Regge-Wheeler equation	37
∇_k	covariant derivative operator	3
Δ	$r^2 - 2Mr$	30
M	mass of Schwarzschild black hole	22
r	space-time radius coordinate	18
$\triangle$	small difference between nearby points on $\mathcal{Y}$	3
δ	small increments	35
l^i, n^i	null-vectors	9
m^i and $\overline{m^i}$	pair of complex conjugate null-vectors	9
κ,ρ,$\varnothing$, ϖ,$\wp$,γ,$\curlywedge$	spin coefficients of l^i,n^i,m^i,$\overline{m^i}$	10
τ,α, ς,π,ϑ		
$\boldsymbol{e}_a$	contravariant basis vectors	9
η_{ab}	tetrad fundamental matrix	9
$\aleph_{abcd}$	Ricci rotation coefficients	10
C_{lijk}	Weyl tensor	7
(t, r, θ, ϕ)	Schwarzschild (spherical) coordinates	18
(v, r, θ, ϕ),	Eddington-Finkelstein coordinates	29
and (u, r, θ, ϕ)		
$\mathcal{I}^+$	future null infinity	14
$\mathcal{I}^-$	past null infinity	27
D,$\Box$,Ξ,Ξ^*	symbols for directional derivative	10
	along l^i,n^i,m^i,$\overline{m^i}$	
G	Gravitational constant	23
$\hat{\rho}$	constant density of the star	23

Symbol	Representation (Bondi-Sachs)	Notation Page
Γ	world tube	14
u	used again to label outgoing null hyper-surface	14
r	used again as surface area coordinate	15
W, β, U^A, and h_{AB}	Bondi metric variables	15
x^A	angular coordinate for the world tube Γ	15
q_{AB}	unit sphere metric	70
q_A	complex dyad	70
U, J	complex metric quantities	70
w	linearized metric quantity	71
$\eth$,$\bar{\eth}$	differential eth operators	70
$\mathcal{V}$	quantity of spin-weight s	70
Y_{lm}	spherical harmonics	62

Chapter 1

Introduction

We start by introducing the following curvature concepts: metric tensor, properties of the Riemann curvature tensor, Ricci curvature tensor and Ricci curvature scalar; and vacuum field equations. These concepts shall be used in later chapters and before then, we need to fully understand these concepts. Secondly we introduce the Weyl tensor since we need to use it when we later introduce the Newmann-Penrose formalism. Thirdly we look at how Einstein vacuum equations are solved using the algebraic computing package called Maple. Thirdly we introduce concepts behind the Newman-Penrose formalism which are important for chapter 2 where we shall be expressing the Schwarzschild geometry in this formalism with the aim of deriving the Bardeen-Press equation in chapter 3, and in chapter's 4 and 5 where we shall be using the Bondi-Sachs metric which is based on this formalism. Lastly we introduce a concise overview of the Bondi-Sachs metric with the aim of providing the reader with the full general understanding of the dynamics of this metric.

1.1 Curvature and Einstein vacuum equations

1.1.1 The metric tensor

We introduce the metric tensor (a symmetric tensor of type (0,2)) because understanding this tensor will help us understand the dynamical information that is contained within it and its mathematical properties.

The metric tensor g_{ij} contains two pieces of information:

1. the information concerning the specific coordinate systems used (e.g., spherical coordinates, Cartesian coordinates, etc.) which is really not important, and

2. the information regarding the existence of any gravitational potentials, which is very important.

The metric tensor provides us with an inner product $g_{ij}\mathbf{A}^i\mathbf{U}^j$ for vectors $\mathbf{A}^i$ and $\mathbf{U}^j$ at each point P of a manifold $(\mathcal{M}, g_{ij})$ [37, 45, 58, 64]. There are four ways of writing this inner product (i.e)

$$g_{ij}\mathbf{A}^i\mathbf{U}^j = g^{ij}\mathbf{A}_i\mathbf{U}_j = \mathbf{A}_i\mathbf{U}^i = \mathbf{A}^i\mathbf{U}_i. \tag{1.1}$$

It is required that an inner product should be positive definite, which means that $g_{ij}\mathbf{A}^i\mathbf{A}^j \geq 0$ for all vectors $\mathbf{A}^i$, with $g_{ij}\mathbf{A}^i\mathbf{A}^j = 0$ only if $\mathbf{A}^i = 0$ [37]. In GR we are required to relax the positive definite condition of the inner product so as to be able to model the space time of general relativity. Hence we only require that the metric tensor be nonsingular, in the sense that matrix $[g_{ij}]$ has an inverse matrix $[g^{ij}]$. This leads to some rather odd metrical properties, such as nonzero vectors having zero length, and the need to include modulus signs where square roots are involved. A manifold that possesses a positive definite metric tensor is called Riemannian, and the one that possesses an indefinite metric tensor is called pseudo-Riemannian (or semi- Riemannian) [37, 58, 64].

Given a curve $\mathcal{Y}$ on a manifold $\mathcal{M}$ which we define by setting $x^a = x^a(\zeta)$, where ζ is the parameter belonging to some interval I, we define at each point on $\mathcal{Y}$ the tangent vector by $\dot{x} \equiv dx^a/d\zeta$. We also define $\triangle s$ to be the distance between nearby points on $\mathcal{Y}$ whose coordinates differences are small. These points are given by the parameter values whose difference $\triangle\zeta$ is small, and to a first order $\triangle x^i = \dot{x}^i \triangle\zeta$, such that $\triangle s^2 = |g_{ij} dx^i dx^j|$, the infinitesimal version is given by

$$ds^2 = |g_{ij} dx^i x^j|, \tag{1.2}$$

which defines the line element of $\mathcal{M}$.

The manifold that we use to model space-time is a four-dimensional pseudo-Riemannian manifold whose metric tensor $g_{\mu\nu}(\mu, \nu = 0, 1, 2, 3)$ has an indefiniteness characterized by $(- + ++)$ [37, 45, 58, 64]. This means that if at any point P we adopt a coordinate system that gives $[g_{\mu\nu}]_P$ as a diagonal matrix, then one of the diagonal elements is negative and the other three are positive. Any non-zero vector is then described as *time-like* if $g_{\mu\nu}\mathbf{A}^\mu\mathbf{A}^\nu < 0$, *null* if $g_{\mu\nu}\mathbf{A}^\mu\mathbf{A}^\nu = 0$ and *spacelike* if $g_{\mu\nu}\mathbf{A}^\mu\mathbf{A}^\nu > 0$.

Also the metric tensor g_{ij} and the contravariant metric tensor g^{ij} have covariant derivatives that are zero given by

$$\begin{aligned} \nabla_k g_{ij} &= 0 \\ \nabla_k g^{ij} &= 0 \end{aligned} \tag{1.3}$$

where ∇ stands for the covariant derivative operator.

1.1.2 Properties of the Riemann curvature tensor

(a) The Riemann curvature tensor

To understand the role of the Riemann tensor in General Relativity (GR), we think of it as representing the gravitational field and it plays a major role in many respects in GR like the one played by the electromagnetic field strength $\mathbf{F}_{ij}$ in Maxwell's theory [3, 47, 51, 60, 78]. It arises as a measure of the extent to which the commutativity fails for the second covariant derivative of a vector field [37]. We best define it implicitly in the following theorem (see [47] for the proof),

Theorem 1.1.1 *Let $\mathcal{M}$ be a manifold with a connection*[1] Γ^k_{ij} *that is at least once differentiable in each coordinate patch. Then there exists a unique tensor field* $R^l{}_{ijk}$ *such that:*

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\boldsymbol{K}_k - T^l_{ij}\nabla_l \boldsymbol{K}_k = -R^l{}_{ijk}\boldsymbol{K}_l \tag{1.4}$$

for any smooth vector field $\boldsymbol{K}_k$ *and a torsion* T^l_{ij} *which is defined by*

$$T^l_{ij} = -\Gamma^l_{ij} + \Gamma^l_{ji} \tag{1.5}$$

One must note that Γ^k_{ij} is not a tensor and $R^l{}_{ijk}$ is given by the following formula [19]

$$R^l{}_{ijk} = \Gamma^l_{ki,j} - \Gamma^l_{ji,k} + \Gamma^l_{jm}\Gamma^m_{ki} - \Gamma^l_{km}\Gamma^m_{ji}. \tag{1.6}$$

But since in GR we deal with only a torsion-free scenario ($T^l_{ij} = 0$), then Eq. (1.4) reduces to

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\mathbf{K}_k = -\frac{1}{2}R^l{}_{ijk}\mathbf{K}_l \tag{1.7}$$

and Eq. (1.6) is still valid.

(b) Symmetries of Riemann curvature tensor

The symmetry $R^l{}_{ijk} = -R^l{}_{jik}$ follows from Eq. (1.6). In the *torsion-free* case, we give the symmetry implicitly in the following proposition [37, 47] without proof.

Proposition 1.1.1 *If* ∇_i *is torsion-free then* $R^l{}_{[ijk]} = 0$*, known as the cyclic identity.*

[1]The name "connection" comes from the fact that Γ^k_{ij} is used to transport vectors from one tangent space to another [19].

We note that $R^l{}_{[ijk]} = \frac{1}{3}(R^l_{ijk} + R^l_{jki} + R^l_{kij})$ which follows from the fact that $R^l{}_{[ijk]}$ is antisymmetric over its first two indices. These symmetries reduces the number of independent components of $R^l{}_{[ijk]}$ from $4 \times 4 \times 4 \times 4 = 256$ to 20 [60, 61].

(c) The Ricci identities

The Ricci identities are important identities because they define the Riemann tensor in terms of the skew-symmetry of second covariant derivative of an arbitrary vector field. From the above discussion we have

$$\nabla_{[i}\nabla_{j]}\mathbf{K}_k = \frac{1}{2}T^l_{ij}\nabla_l\mathbf{K}_k - \frac{1}{2}R^k{}_{ijk}\mathbf{K}_k, \tag{1.8}$$

for covariant vectors. Now by the use of

$$\nabla_{[i}\nabla_{j]}E = \frac{1}{2}T^k_{ij}\nabla_k E \tag{1.9}$$

where $E = \mathbf{K}_k\mathbf{V}^k$, we are able to get the result for contravariant vectors given by

$$\nabla_{[i}\nabla_{j]}\mathbf{V}^l = \frac{1}{2}T^k_{ij}\nabla_k\mathbf{V}^l + \frac{1}{2}R^l{}_{ijk}\mathbf{V}^k, \tag{1.10}$$

where $\mathbf{V}^l$ is a smooth vector field.

(d) The Bianchi identities

In analogy to the electromagnetic field tensor F_{ij} which satisfy the condition $\nabla_{[i}F_{jk]} = 0$ (whether or not the vacuum condition applies), the Riemann tensor happens to satisfy a similar condition.

To substantiate the above statement we provide with the following proposition [47] without proof

Proposition 1.1.2 *If ∇_i is torsion-free them $\nabla_{[i}R^l{}_{jk]m} = 0$.*

The Bianchi identity hold generally for any torsion-free connection, and it is of great importance in GR.

1.1.3 The Ricci curvature tensor and the Ricci curvature scalar

We defined two more basic tensors in terms of curvature tensor $R^l{}_{ijk}$. These are Ricci tensor and Ricci scalar.
We define the Ricci tensor R_{ij} as

$$R_{ij} = R^l{}_{ilj} = R_{ji}, \tag{1.11}$$

and it is given by the following equation [19, 37, 80]

$$R_{ij} = \Gamma^l_{li,j} - \Gamma^l_{ij,l} + \Gamma^k_{il}\Gamma^l_{jk} - \Gamma^k_{ij}\Gamma^l_{kl}. \tag{1.12}$$

It is the contraction of R^l_{ikj} on the first and third indices. Other contractions would in principle also be possible i.e. on the first and second, the first and fourth, etc. But because R_{ijlk} is antisymmetric on i and j and on l and k, all these contractions either vanish or reduce to $\pm R_{ij}$. Therefore the Ricci tensor is essentially the only contraction of the Riemann tensor and it has 10 independent components. The Ricci scalar is defined as [19, 37, 80]

$$R = g^{ij}R_{ij} = g^{ij}g^{lk}R_{likj}. \tag{1.13}$$

1.1.4 The Einstein vacuum equations

From Eqs. (1.11) and (1.13) we have the following proposition:

Proposition 1.1.3 $\nabla^i(R_{ij} - \frac{1}{2}g_{ij}R) = 0.$

Proof. We begin with the Bianchi identity $\nabla_{[i}R_{jk]lm} = 0$, that is

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.$$

Then contracting this relation with $g^{il}g^{km}$ we get

$$2\nabla^i R_{ij} - \nabla_j R = 0,$$

from which the result follows.

We define the Einstein tensor G_{ij} to be $G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$ and it is found through the contracted Bianchi identity to be divergence-free: $\nabla^i G_{ij} = 0$. This results is of fundamental significance, and a primary basis for the arguments leading to Einstein vacuum equations. The Einstein vacuum equations (i.e. the gravitational field equations for region of space in the absent of matter) are given by $R_{ij} = 0$ and one must note that $R_{ij} = 0$ if and only if $G_{ij} = 0$.

1.2 The Weyl tensor

As we have seen in the previous section that in 4-dimensions, $R^l{}_{[ijk]}$ has 20 independent components and R_{ij} has 10. This means that we should be able to decompose $R^l{}_{[ijk]}$ into terms consisting of R_{ij} plus some other tensor which has 10 independent components as follows

$$\begin{aligned} R_{lijk} &= \frac{1}{2}(g_{lj}R_{ik} - g_{li}R_{ij} - g_{ij}R_{lk} + g_{ik}R_{lj}) && (1.14)\\ &- \frac{1}{6}R(g_{lj}g_{ik} - g_{lk}g_{ij}) + C_{lijk}. && (1.15)\end{aligned}$$

where C_{lijk} is the Weyl tensor.

The above equation is designed in such a way that all possible contraction of C_{lijk} vanish, while it retains the symmetry properties of the Riemann tensor as follows

$$C_{lijk} = C_{[li][jk]}, \tag{1.16}$$

$$C_{lijk} = C_{jkli}, \tag{1.17}$$

$$C_{l[ijk]} = 0 \tag{1.18}$$

The most important property of the Weyl tensor is that it is invariant under conformal transformations. This means that if you compute C_{lijk} for some metric g_{jk}, and then compute it again for a metric given by $\Omega^2(x)g_{jk}$, where $\Omega(x)$ is an arbitrary nonvanishing function of space-time, you get the same answer. Because of this property it is often referred to as the *conformal tensor*.

1.3 Solving Einstein vacuum field equations

The field equations of GR are extremely complex set of ten coupled, nonlinear, hyperbolic-elliptic partial differential equations that are not solvable in many situations of interest. In fact, state of the art computers might not be able to evolve solutions far enough in time for the physics of interest to be captured. Yet, if one assumes spherical symmetry, one would arrive at an exact solution which has been known since 1916 [81]. The exact solution is the Schwarzschild solution that describes the Schwarzschild geometry of a black hole and we shall derive it in chapter 2.

Einstein himself thought that it would be impossible to find an exact solution of these equations (see [60] page. 431-434). It was precisely for this reason that algebraic computing first came into existence. In the words of Jean Sammett in her article on this emerging field published in 1966 [75],

"*It has become obvious that there are a large number of problems requiring very tedious, time-consuming, error-prone and straightforward algebraic manipulation, and these characteristics make computer solutions both necessary and desirable*".

Algebraic computing is the field of using computers for carrying out algebraic calculations. Most researchers in this field preferrer to use general purpose systems like REDUCE, MATHEMATICA, MAPLE and MACSYMA. MAPLE was used by Bishop [13] to obtain the results of chapter 4. In this dissertation we shall use MAPLE to do all the algebraic computations of chapter 5. We have also included MAPLE programs in the Appendixes A, B, C, and E.

1.4 Newman-Penrose formalism

The Newman-Penrose formalism was developed to introduce spinor calculus into general relativity. It is a special instance of tetrad calculus with a special choice of the basis vectors. The resulting choice that is made, is a tetrad of null vectors $\{l^i, n^i, m^i, \overline{m^i}\}$ of which l^i and n^i are real and m^i and $\overline{m^i}$ are complex conjugates of one another. The light-cone structure of the space-times of black-hole solutions of general relativity is of the kind that makes the Newman-Penrose formalism most effective for grasping the inherent symmetries of these space-times and revealing their analytic richness [25]. But the most special adaptability of the Newman-Penrose formalism to the black-hole solutions of general relativity derives from their "type-D" character and the Goldberg-Sachs theorem [25]. We now present a brief summery of the basic equations that we shall use in chapters 3 and 5.

The pair of real null-vectors l^i and n^i and a pair of complex conjugate null-vectors m^i and $\overline{m^i}$ are required to satisfy the *orthogonality conditions*

$$l^i{\cdot}m_i = l^i{\cdot}\overline{m_i} = n^i{\cdot}m_i = n^i{\cdot}\overline{m_i} = 0, \tag{1.19}$$

and the null requirements

$$l^i{\cdot}l_i = n^i{\cdot}n_i = m^i{\cdot}m_i = m^i \cdot \overline{m_i} = 0. \tag{1.20}$$

We further impose on the basis vectors, the following normalization conditions

$$l^i{\cdot}n_i = 1 \qquad \textit{and} \qquad m^i{\cdot}\overline{m_i} = -1, \tag{1.21}$$

which imply that the fundamental matrix represented by η_{ab} is a constant symmetric matrix of the form

$$[\eta_{ab}] = [\eta^{ab}] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{1.22}$$

We define the contravariant basis vectors $\boldsymbol{e}_a$ as

$$\boldsymbol{e}_1 = l^i, \quad \boldsymbol{e}_2 = n^i, \quad \boldsymbol{e}_3 = m^i, \quad \boldsymbol{e}_4 = \overline{m^i} \tag{1.23}$$

and the corresponding covariant basis as

$$\boldsymbol{e}^1 = \boldsymbol{e}_2 = n^i, \qquad \boldsymbol{e}^2 = \boldsymbol{e}_1 = l^i, \qquad \boldsymbol{e}^3 = -\boldsymbol{e}_4 = -\overline{m^i}, \qquad \boldsymbol{e}^4 = -\boldsymbol{e}_3 = -m^i. \tag{1.24}$$

From now on, we designate the basis vectors considered as directional derivatives by the symbols: D, $\Box$, Ξ, and Ξ^* as followers

$$\boldsymbol{e}_1 = \boldsymbol{e}^2 = D; \quad \boldsymbol{e}_2 = \boldsymbol{e}^1 = \Box; \tag{1.25}$$

$$\boldsymbol{e}_3 = -\boldsymbol{e}^4 = \Xi; \quad \boldsymbol{e}_4 = -\boldsymbol{e}^3 = \Xi^*. \tag{1.26}$$

The basic quantities of the formalism are the *spin coefficients*, of which there are twelve complex ones:

$$\begin{array}{llll}
\kappa = \aleph_{311}; & \rho = \aleph_{314}; & \varnothing = \frac{1}{2}(\aleph_{211} + \aleph_{341}); \\
\varpi = \aleph_{313}; & \wp = \aleph_{243}; & \gamma = \frac{1}{2}(\aleph_{212} + \aleph_{342}); \\
\curlywedge = \aleph_{244}; & \tau = \aleph_{312}; & \alpha = \frac{1}{2}(\aleph_{214} + \aleph_{344}); \\
\varsigma = \aleph_{242}; & \pi = \aleph_{241}; & \vartheta = \frac{1}{2}(\aleph_{213} + \aleph_{343}).
\end{array} \tag{1.27}$$

where

$$\aleph_{abcd} = e_c{}^k e_{ak;i} e_b{}^i \quad \text{(Ricci rotation coefficients)}. \tag{1.28}$$

The whole set of field equations in the formalism come by writing the Ricci and Bianchi identities using these coefficients, and they take the place of the Einstein equations.

The ten components of the Ricci tensor are defined in terms of the following four real and three complex scalars:

$$\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{11}; \quad \Phi_{22} = -\frac{1}{2}R_{22}; \quad \Phi_{02} = -\frac{1}{2}R_{33}; \quad \Phi_{20} = -\frac{1}{2}R_{44}; \\
\Phi_{11} &= -\frac{1}{4}(R_{12} + R_{34}); \quad \Phi_{01} = -\frac{1}{2}R_{13}; \quad \Phi_{10} = -\frac{1}{2}R_{14}; \\
\Lambda &= \frac{1}{24}R = \frac{1}{12}(R_{12} - R_{34}); \quad \Phi_{12} = -\frac{1}{2}R_{23}; \quad \Phi_{21} = -\frac{1}{2}R_{24}.
\end{aligned} \tag{1.29}$$

All ten independent components of the Weyl tensor C_{pqrs} can be written as five complex scalars:

$$\begin{aligned}
\Psi_0 &= -C_{1313} = -C_{pqrs} l^p m^q l^r m^s, \\
\Psi_1 &= -C_{1213} = -C_{pqrs} l^p n^q l^r m^s, \\
\Psi_2 &= -C_{1342} = -C_{pqrs} l^p m^q \overline{m}^r n^s, \\
\Psi_3 &= -C_{1242} = -C_{pqrs} l^p n^q \overline{m}^r n^s, \\
\Psi_4 &= -C_{2424} = -C_{pqrs} n^p \overline{m}^q n^r \overline{m}^s.
\end{aligned} \tag{1.30}$$

where the tetrad components of the Weyl tensor are given by

$$C_{pqrs} = R_{pqrd} + \frac{1}{2}(\eta_{pr}R_{qs} - \eta_{qr}R_{ps} - \eta_{ps}R_{qr} + \eta_{bd}R_{pr}) + \frac{1}{6}(\eta_{pr}\eta_{qs} - \eta_{ps}\eta_{qr})R. \tag{1.31}$$

The Ricci identities are given by:

$$\begin{aligned}
D\rho - \Xi^*\kappa &= (\rho^2 + \varpi\varpi^*) + \rho(\varnothing + \varnothing^*) \\
&- \kappa^*\tau - \kappa(3\alpha + \vartheta^* - \pi) + \Phi_{00}, && [R_{1314}] \quad (1.32) \\
D\varpi - \Xi\kappa &= \varpi(\rho + \rho^* + 3\varnothing - \varnothing^*) \\
&- \kappa(\tau - \pi^* + \alpha^* + 3\vartheta) + \Psi_0, && [R_{1313}] \quad (1.33) \\
D\tau - \Box\kappa &= \rho(\tau + \pi^*) + \varpi(\tau^* + \pi) + \tau(\varnothing - \varnothing^*) \\
&- \kappa(3\gamma + \gamma^*) + \Psi_1 + \Phi_{01}, && [R_{1312}] \quad (1.34) \\
D\alpha - \Xi^*\varnothing &= \alpha(\rho + \varnothing^* - 2\varnothing) + \vartheta\varpi^* - \vartheta^*\varnothing - \kappa \curlywedge \\
&- \kappa^*\gamma + \pi(\varnothing + \rho) + \Phi_{10}, && [\tfrac{1}{2}(R_{3414} - R_{1214})] \quad (1.35) \\
D\vartheta - \Xi\varnothing &= \varpi(\alpha + \pi) + \vartheta(\rho^* - \varnothing^*) - \kappa(\wp + \gamma) \\
&- \varnothing(\alpha^* - \pi^*) + \Psi_1, && [\tfrac{1}{2}(R_{1213} - R_{3413})] \quad (1.36) \\
D\gamma - \Box\varnothing &= \alpha(\tau + \pi^*) + \vartheta(\tau^* + \pi) - \gamma(\varnothing + \varnothing^*) \\
&- \varnothing(\gamma + \gamma^*) + \tau\pi - \varsigma\kappa + \Psi_2 + \Phi_{11} - \Lambda, && [\tfrac{1}{2}(R_{1212} - R_{3412})] \quad (1.37) \\
D \curlywedge - \Xi^*\pi &= (\rho \curlywedge + \varpi^*\wp) + \pi(\pi + \alpha - \vartheta) - \varsigma\kappa^* \\
&- \curlywedge(3\varnothing - \varnothing^*) + \Phi_{20}, && [R_{2441}] \quad (1.38)
\end{aligned}$$

$$
\begin{aligned}
D\wp - \Xi\pi &= (\rho^*\wp + \varpi\curlywedge) + \pi(\pi^* - \alpha^* + \vartheta) \\
&- \wp(\varnothing + \varnothing^*) - \varsigma\kappa + \Psi_2 + 2\Lambda, && [R_{2431}] \quad (1.39) \\
D\varsigma - \Box\pi &= \wp(\pi + \tau^*) + \curlywedge(\pi^* + \tau) + \pi(\gamma - \gamma^*) \\
&- \varsigma(3\varnothing + \varnothing^*) + \Psi_3 + \Phi_{21}, && [R_{2421}] \quad (1.40) \\
\Box \curlywedge - \Xi^*\varsigma &= -\curlywedge(\wp + \wp^* + 3\gamma - \gamma^*) \\
&+ \varsigma(3\alpha + \vartheta^* + \pi - \tau^*) - \Psi_4, && [R_{2442}] \quad (1.41) \\
\Xi\rho - \boxtimes^*\varpi &= \rho(\alpha^* + \vartheta) - \varpi(3\alpha - \vartheta^*) + \tau(\rho - \rho^*) \\
&+ \kappa(\wp - \wp^*) - \Psi_1 + \Phi_{01}, && [R_{3143}] \quad (1.42) \\
\delta\alpha - \Xi^*\vartheta &= (\wp\rho - \curlywedge\varpi) + \alpha\alpha^* + \vartheta\vartheta^* - 2\alpha\xi \\
&+ \gamma(\rho - \rho^*) + \varnothing(\wp - \wp^*) - \Psi_2 + \Phi_{11} + \Lambda, && [\tfrac{1}{2}(R_{1234} - R_{3434})] \quad (1.43) \\
\Xi \curlywedge - \Xi^*\wp &= \varsigma(\rho - \rho^*) + \pi(\wp - \wp^*) + \wp(\alpha + \vartheta^*) \\
&+ \curlywedge(\alpha^* - 3\vartheta) - \Psi_3 + \Phi_{21}, && [R_{2443}] \quad (1.44) \\
\Xi\varsigma - \Box\wp &= (\wp^2 + \curlywedge\curlywedge^*) + \wp(\gamma + \gamma^*) - \varsigma^*\pi \\
&+ \varsigma(\tau - 3\vartheta - \alpha^*) + \Phi_{22}, && [R_{2423}] \quad (1.45) \\
\Xi\gamma - \Box\vartheta &= \gamma(\tau - \alpha^* - \vartheta) + \wp\tau - \varpi\nu - \varnothing\varsigma^* \\
&- \vartheta(\gamma - \gamma^* - \mu) + \alpha\curlywedge^* + \Phi_{12}, && [(\tfrac{1}{2}R_{1232} - R_{3432})] \quad (1.46) \\
\Xi\tau - \Box\varpi &= (\wp\varpi + \curlywedge^*\rho) + \tau(\tau + \vartheta - \alpha^*) \\
&- \varpi(3\gamma - \gamma^*) - \kappa\nu^* + \Phi_{02}, && [R_{1332}] \quad (1.47) \\
\Box\rho - \Xi^*\tau &= -(\rho\wp^* + \varpi\curlywedge) + \tau(\vartheta^* - \alpha - \tau^*) \\
&+ \rho(\gamma + \gamma^*) + \varsigma\kappa - \Psi_2 - 2\Lambda, && [R_{1324}] \quad (1.48) \\
\Box\alpha - \Xi^*\gamma &= \varsigma(\rho + \varnothing) - \curlywedge(\tau + \vartheta) + \alpha(\gamma^* - \wp^*) \\
&+ \gamma(\vartheta^* - \tau^*) - \Psi_3. && [\tfrac{1}{2}(R_{1242} - R_{3442})] \quad (1.49)
\end{aligned}
$$

The eight components of the Bianchi identities are given explicitly by:

$$
\begin{aligned}
&- \Xi^*\Psi_0 + D\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \varnothing)\Psi_1 + 3\kappa\Psi_2 \\
&+ [Ricci] = 0; \qquad R_{12[13|4]} = 0, \qquad (1.50) \\
&+ \Xi^*\Psi_1 - D\Psi_2 - \curlywedge\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad [Ricci] = 0; && R_{13[21|4]} = 0, && (1.51)\\
&- \quad \Xi^*\Psi_2 + D\Psi_3 + 2 \curlywedge \Psi_1 - 3\pi\Psi_2 + 2(\varnothing - \rho)\Psi_3 + \kappa\Psi_4 && && \\
&+ \quad [Ricci] = 0; && R_{42[13|4]}, && (1.52)\\
&+ \quad \Xi^*\Psi_3 - D\Psi_4 - 3 \curlywedge \Psi_2 + 2(2\pi + \alpha)\Psi_3 - (4\varnothing - \rho)\Psi_4 && && \\
&+ \quad [Ricci] = 0; && R_{42[21|4]} = 0, && (1.53)\\
&+ \quad \Box\Psi_0 + \Xi\Psi_1 + (4\gamma - \wp)\Psi_0 - 2(2\tau + \vartheta)\Psi_1 + 3\varpi\Psi_2 && && \\
&+ \quad [Ricci] = 0; && R_{13[13|2]} = 0, && (1.54)\\
&- \quad \Box\Psi_1 + \Xi\Psi_2 + \varsigma\Psi_0 + 2(\gamma - \wp)\Psi_1 - 3\tau\Psi_2 + 2\varpi\Psi_3 && && \\
&+ \quad [Ricci] = 0; && R_{13[43|2]} = 0, && (1.55)\\
&- \quad \Box\Psi_2 + \Xi\Psi_3 + 2\varsigma\Psi_1 - 3\wp\Psi_2 + 2(\vartheta - \tau)\Psi_3 + \varpi\Psi_4 && && \\
&+ \quad [Ricci] = 0; && R_{42[13|2]} = 0, && (1.56)\\
&- \quad \Box\Psi_3 + \Xi\Psi_4 + 3\varsigma\Psi_2 - 2(\gamma + 2\wp)\Psi_3 - (\tau - 4\vartheta)\Psi_4 && && \\
&+ \quad [Ricci] = 0; && R_{42[43|2]} = 0, && (1.57)
\end{aligned}
$$

where the terms in the Ricci tensors (enclosed in square brackets i.e $[Ricci]$), in the respective equations, are:

$$
\begin{aligned}
&- && D\Phi_{01} + \Xi\Phi_{00} + 2(\varnothing + \rho^*)\Phi_{01} + 2\varpi\Phi_{10} - 2\kappa\Phi_{11} - \kappa^*\Phi_{02} \\
&+ && (\pi^* - 2\alpha^* - 2\vartheta)\Phi_{00}, && (1.58)\\
&+ && \Xi^*\Phi_{01} - \Box\Phi_{00} - 2(\alpha + \tau^*)\Phi_{01} + 2\rho\Phi_{11} + \varpi^*\Phi_{02} \\
&- && (\wp^* - 2\gamma - 2\gamma^*)\Phi_{00} - 2\tau\Phi_{10} - 2D\Lambda, && (1.59)\\
&- && D\Phi_{21} + \Xi\Phi_{20} + 2(\rho^* - \varnothing)\Phi_{21} - 2\wp\Phi_{10} + 2\pi\Phi_{11} - \kappa^*\Phi_{22} \\
&- && (2\alpha^* - 2\vartheta - \pi^*)\Phi_{20} - 2\Xi^*\Lambda, && (1.60)\\
&- && \Box\Phi_{20} + \Xi^*\Phi_{21} + 2(\alpha - \tau^*)\Phi_{21} + 2\varsigma\Phi_{10} + \varpi^*\Phi_{22} - 2 \curlywedge \Phi_{11} \\
&- && (\wp^* + 2\gamma - 2\gamma^*)\Phi_{20}, && (1.61)\\
&- && D\Phi_{20} + \Xi\Phi_{01} + 2(\pi^* - \vartheta)\Phi_{01} - 2\kappa\Phi_{12} - \curlywedge^*\Phi_{00} + 2\varpi\Phi_{11} \\
&+ && (\rho^* + 2\varnothing - 2\varnothing^*)\Phi_{02}, && (1.62)\\
&\Box\Phi_{01} && -\Xi^*\Phi_{02} + 2(\wp^* - \gamma)\Phi_{01} - 2\rho\Phi_{12} - \varsigma^*\Phi_{00} + 2\tau\Phi_{11}
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & (\tau^* - 2\vartheta^* + 2\alpha)\Phi_{02} + 2\Xi\Lambda, & (1.63)\\
- \quad & D\Phi_{22} + \Xi\Phi_{21} + 2(\pi^* + \vartheta)\Phi_{21} - 2\wp\Phi_{11} - \curlywedge^*\Phi_{20} + 2\pi\Phi_{12} & \\
+ \quad & (\rho^* - 2\varnothing - 2\varnothing^*)\Phi_{22} - 2\Box\Lambda, & (1.64)\\
\Box\Phi_{21} \quad & -\Xi^*\Phi_{22} + 2(\wp^* + \gamma)\Phi_{21} - 2\varsigma\Phi_{11} - \varsigma^*\Phi_{20} + 2\curlywedge\Phi_{12} & \\
+ \quad & (\tau^* - 2\alpha - 2\vartheta^*)\Phi_{22}. & (1.65)
\end{aligned}
$$

To do perturbation calculations one specifies the perturbed geometry by introducing slight changes in the tetrad like $l^i = l^A + l^B$, $n^A + n^B$, etc. Here the terms with superscript A are the unperturbed values and the ones with superscript B the small perturbation. Then, all the Newmann-Penrose spin coefficients and other quantities can also be written in a similar fashion: $\Psi_4 = \Psi_4^A + \Psi_4^B$, etc. We obtain the linear perturbation equations by keeping the terms with superscript B only up to first order.

1.5 An overview of the Bondi-Sachs metric

When a Schwarzschild black hole is perturbed, it produces gravitational radiation which travels to future null infinity ($\mathcal{I}^+$) [84]. Numerical relativity has been used to unveil the dynamics of gravitational radiation from a Schwarzschild black hole and black holes in general. Over the last few years we have witnessed an important and exciting development of applying linear perturbation theory of black holes to aid in the verification and interpretation of numerical simulations of gravitational radiation from a Schwarzschild back hole [82].

In a series of papers Bondi and others [16, 62, 65, 73, 86] developed the theory of gravitational radiation which yields the description of the "plus" and "cross" polarization modes of gravitational radiation in terms of the real and imaginary parts of the Bondi news function at $\mathcal{I}^+$. Bondi's initial use of null coordinates to describe radiation field [15] was followed by a rapid development of other null formalisms [98]. These were differentiated either as metric based approaches, as developed for axisymmetric isolated systems by Bondi, Metzner and van den Burg [16] and generalized by Sachs [73], or as null-tetrad approaches in which the Bianchi identities appear as part of the set of equations, as developed by Newman and Penrose [62].

At the core of this theory is the Bondi-Sachs metric [6, 9, 10, 11, 13, 12]

$$\begin{aligned} ds^2 = & - \left(e^{2\beta}(1+\frac{W}{r}) - r^2 h_{AB} U^A U^B \right) du^2 - 2e^{2\beta} du dr \\ & - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B, \end{aligned} \tag{1.66}$$

where W, β, U^A, and h_{AB} are free metric variables and
u labels the outgoing null hypersurface,
x^A are angular coordinates (the null rays emanating from the world tube Γ [2] which is used as the inner boundary of a Characteristic foliation [16, 73]),
r is the surface area coordinate,
h_{AB} is the conformal metric (angular metric) which contains 2 radiative degrees of freedom,
β is the expansion of the light rays propagating outwards and $e^{2\beta}$ measures the expansion of the null cone.

For a Schwarzschild black hole we set $W = -2M$, M is the mass of a Schwarzschild black hole, as well as $U^\beta = 0$, $\beta = 0$, h_{AB} a unit sphere metric. Eq. (1.66) may be evolved numerically by means of three hypersurface equations and the evolution equations [98]. These equations form a hierarchical set [13], meaning that they can be integrated radially to determine β, U, W on the hypersurface in that order in terms of the integration constants determined by the boundary conditions, or smoothness if extended to the vertex of null cone [98]. In chapter 5, we shall transform Chandrasekhar's results on even and odd-parity perturbations [25] to Bondi-Sachs form.

1.6 Conclusion

We introduced curvature, Einstein vacuum field equations and looked at ways of solving Einstein field equations using Maple. We also introduced the Newman-Penrose

[2] Γ must not be confused with the connection Γ^k_{ij}.

formalism and a concise overview of the Bondi-Sachs metric. We shall use these concepts in later chapters. Next we introduce the concept of Schwarzschild space time.

Chapter 2

Schwarzschild space-time

2.1 Introduction

The aim of this chapter is to discuss the geometrical aspects of a Schwarzschild space-time so that we can fully understand the geometrical structure of a Schwarzschild black hole. Hence, in this chapter we introduce the concept of spherically symmetric space-times followed by the derivation of Schwarzschild solution with reference to the Birkhorff's theorem. We then discussion Schwarzschild geometry and its Carter-Penrose diagrams and causal properties, and lastly we describe Schwarzschild geometry in the Newman-Penrose formalism so that we can derive the Bardeen-Press equation later on.

2.2 Spherically symmetric space-times

A spherically symmetric space-time may be loosely defined as one that admits a preferred timelike observer such that the space-time is spherically symmetric about every point on this special observer's world-line.[1] One can then prove the theorem [34, 79] that says a spherically symmetric space-time is the direct product $\mathcal{M} = S^2 \times N$, where S^2 is the 2-sphere manifold with the standard metric g_S on the unit sphere and

[1]Spherically symmetric space-time is defined as one which admits the group $SO(3)$ as a group of isometries, with the group orbits space-like two-surface [78]

N is a 2-dimensional manifold with a Lorentzian (indefinite) metric g_N, and with a scalar r such that the complete space-time metric g_{ij} is "conformally decomposable", i.e. $r^{-2}g_{ij}$ is the direct sum of the 2-dimensional parts g_N and g_S. Leaving further technicalities aside (see [35, 79]) we then write down the final spherically symmetric line element in the form

$$ds^2 = -e^{2\varsigma}dt^2 + e^{2\wp_2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

where we permit $\varsigma(r,t)$ and $\wp_2(r,t)$ to have an imaginary part $i\pi/2$ so that the signs of dt^2 and dr^2 in Eq. (2.1) and thus the role of r and t as space-time coordinates may interchange [35]. r is defined invariantly by the area $4\pi r^2$ of the 2-spheres $r =$ constant, $t =$ constant. Lastly we note that there is no important relation between r and the proper distance from the center (if there is one) to the spherical surface. Later on in chapter 3 we shall look at the more general version of Eq. (2.1) when introducing Ricci and Einstein tensors for non-stationary axisymmetric space-times.

2.3 Schwarzschild solution

We shall derive a solution of Einstein vacuum equations (see chapter 1, cec. 1.1.4) corresponding to the gravitational field outside an isolated spherically symmetric static body.

We take spherical coordinates,

$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi) \tag{2.2}$$

Spherical symmetry then implies

$$g_{02} = g_{03} = g_{12} = g_{13} = g_{23} = 0, \tag{2.3}$$

as well as

$$g_{33} = \sin^2\theta g_{22}, \tag{2.4}$$

and time-reversal symmetry

$$g_{01} = 0. \tag{2.5}$$

The metric tensor is now specified by writing down the length ds of the infinitesimal line element:

$$ds^2 = -Adt^2 + Bdr^2 + Cr^2(d\theta^2 + r^2 \sin^2\theta d\phi^2), \tag{2.6}$$

where A, B, and C are positive functions depending only on r. At large distance from the source (star) we expect,

$$r \to \infty; \qquad A, B, C \to 1. \tag{2.7}$$

We are now free to choose the new r coordinate:

$$\widetilde{r} = \sqrt{C(r)}r, \quad \text{so that} \quad Cr^2 = \widetilde{r}^2, \tag{2.8}$$

and we now have

$$Bdr^2 = B\left(\sqrt{C} + \frac{r}{2\sqrt{C}}\frac{dC}{dr}\right)^{-2} d\widetilde{r}^2 = \widetilde{B}d\widetilde{r}^2. \tag{2.9}$$

In the new coordinate one has (omitting the tilde$\widetilde{\ }$):

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.10}$$

where A$, B \to 1$ as $r \to \infty$. The signature of this metric must be (-, +, +, +), so that

$$A > 0 \quad \text{and} \quad B > 0. \tag{2.11}$$

Now we need to find the Christoffel symbols Γ for general A and B. If we know all geodesics

$$\ddot{x}^i + \Gamma^i{}_{kl}\dot{x}^k\dot{x}^l = 0, \tag{2.12}$$

then they uniquely determine all Γ coefficients. The variational principle for a geodesic is

$$0 = \delta\int ds = \delta\int\sqrt{g_{ij}\frac{dx^i}{d\zeta}\frac{dx^j}{d\zeta}}d\zeta, \tag{2.13}$$

where ζ is an arbitrary parametrization of the curve. The original curve is chosen to have

$$\zeta = s. \tag{2.14}$$

The square root is then one and then we have

$$\frac{1}{2}\delta\int g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=0. \tag{2.15}$$

and we then write

$$-A\dot{t}^2+B\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2=F(s); \qquad \delta\int F ds=0. \tag{2.16}$$

The dot stands for differentiation with respect to s. Eq. (2.16) generates the Lagrange equation

$$\frac{d}{ds}\frac{\partial F}{\partial\dot{x}^i}=\frac{\partial F}{\partial x^i}. \tag{2.17}$$

For $i=0$ this is

$$\frac{d}{ds}(-2A\dot{t})=0, \tag{2.18}$$

or

$$\ddot{t}+\frac{1}{A}\left(\frac{\partial A}{\partial r}\cdot\dot{r}\right)\dot{t}=0. \tag{2.19}$$

comparing with Eq. (2.12) we see that ${\Gamma^0}_{ij}$ vanish except

$${\Gamma^0}_{10}={\Gamma^0}_{01}=A'/2A \tag{2.20}$$

The accent $'$ stands for differentiation with respect to r; the 2 comes from symmetrization of the subscript indices 0 and 1.

For $i=1$, Eq. (2.17) implies

$$\ddot{r}+\frac{B'}{2B}\dot{r}^2+\frac{A'}{2B}\dot{t}^2-\frac{r}{B}\dot{\theta}^2-\frac{r}{B}\sin^2\theta\dot{\phi}^2=0, \tag{2.21}$$

so that all ${\Gamma^1}_{ij}$ are zero except

$$\begin{aligned}{\Gamma^1}_{00} &= A'/2B \quad ; \quad {\Gamma^1}_{11}=B'/2B;\\ {\Gamma^1}_{22} &= -r/B \quad ; \quad {\Gamma^1}_{33}=-(r/B)\sin^2\theta,.\end{aligned} \tag{2.22}$$

For $i=2$ and 3 we find similarly:

$$\begin{aligned}{\Gamma^2}_{21} &= {\Gamma^2}_{12}=1/r \quad ; \quad {\Gamma^2}_{33}=-\sin\theta\cos\theta;\\ {\Gamma^3}_{23} &= {\Gamma^3}_{32}=\cot\theta \quad ; \quad {\Gamma^3}_{13}={\Gamma^3}_{31}=1/r.\end{aligned} \tag{2.23}$$

Furthermore we have

$$\sqrt{-g} = r^2 \sin\theta \sqrt{AB}. \tag{2.24}$$

and from Eq. (2.18)

$$\Gamma^i{}_{ik} = (\partial_k \sqrt{-g})/\sqrt{-g} = \partial_k \log\sqrt{-g}. \tag{2.25}$$

Therefore

$$\begin{aligned} \Gamma^i{}_{i1} &= A'/2A + B'/2B + 2/r, \\ \Gamma^i{}_{i2} &= \cot\theta. \end{aligned} \tag{2.26}$$

The Ricci tensor

$$R_{ij} = 0, \tag{2.27}$$

now becomes

$$R_{ij} = -(\log\sqrt{-g})_{,i,j} + \Gamma^k{}_{ij,k} - \Gamma^p{}_{ki}\Gamma^k{}_{p,j} + \Gamma^k{}_{ij}(\log\sqrt{-g})_{,k} = 0. \tag{2.28}$$

and then

$$\begin{aligned} R_{00} &= \Gamma^1{}_{00,1} - 2\Gamma^1{}_{00}\Gamma^0{}_{01} + \Gamma^1{}_{00}(\log\sqrt{-g})_{,1} \\ &= (A'/2B)' - A'^2/2AB + (A'/2B)\left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}\right) \\ &= \frac{1}{2B}\left(A'' - \frac{A'B'}{2B} - \frac{A'^2}{2A} + \frac{2A'}{r}\right) = 0, \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} R_{11} = &- (\log\sqrt{-g})_{,1,1} + \Gamma^1{}_{11,1} - \Gamma^0{}_{10}\Gamma^0{}_{10} - \Gamma^1{}_{11}\Gamma^1{}_{11} \\ &- \Gamma^2{}_{21}\Gamma^2{}_{21} - \Gamma^3{}_{31}\Gamma^3{}_{31} + \Gamma^1{}_{11}(\log\sqrt{-g})_{,1} = 0 \end{aligned} \tag{2.30}$$

Then we have

$$\frac{1}{2A}\left(-A'' + \frac{A'B'}{2B} + \frac{A'^2}{2A} + \frac{2AB'}{rB}\right) = 0. \tag{2.31}$$

Combining Eq. (2.29) and Eq. (2.31) we obtain

$$\frac{2}{rB}(AB)' = 0. \tag{2.32}$$

Therefor AB=constant. Since at $r \to \infty$ we have A and $B \to 1$ we conclude that

$$B = 1/A. \tag{2.33}$$

In the $\theta\theta$ direction one has

$$\begin{aligned} R_{22} = & - & (\log\sqrt{-g})_{,2,2} + \Gamma^1{}_{22,1} - 2\Gamma^1{}_{22}\Gamma^2{}_{21} \qquad (2.34) \\ & - & \Gamma^3{}_{23}\Gamma^3{}_{23} + \Gamma^1{}_{22}(\log\sqrt{-g})_{,1} = 0. \qquad (2.35) \end{aligned}$$

This then becomes

$$R_{22} = -\frac{\partial}{\partial\theta}\cot\theta - (\frac{r}{B})' + \frac{2}{B} - \cot^2\theta - \frac{r}{B}\left(\frac{2}{r} + \frac{(AB)'}{2AB}\right) = 0. \tag{2.36}$$

Then using Eq. (2.32) we obtain

$$(r/B)' = 1. \tag{2.37}$$

Upon integration,

$$\begin{aligned} r/B &= r - 2M, \qquad (2.38) \\ A &= 1 - \frac{2M}{r}; \quad B = (1 - \frac{2M}{r})^{-1}. \qquad (2.39) \end{aligned}$$

where $2M$ is an integration constant.

We found the solution even though we did not yet use all equations $R_{ij} = 0$ available to us and only a linear combination of R_{00} and R_{11} was used. It is not hard to see that all the $R_{ij} = 0$ equations are satisfied, first by substituting Eq. (2.39) in Eq. (2.29) or Eq. (2.31), and then spherical symmetry with Eq. (2.36) will also ensure that $R_{33} = 0$.

Finally, substituting Eq. (2.39) in to Eq. (2.10) we get the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.40}$$

2.4 Birkhoff's theorem

Birkhoff's theorem states: *A spherically symmetric gravitational field in empty space must be static(i.e time independent) with a metric given by the Schwarzschild solution.*

The external field of any electrically neutral, spherical star satisfies the conditions of Birkhoff's theorem, whether the star is static, vibrating, or collapsing. A consequence of Birkhoff's theorem is that a radially pulsating distribution of mass can emit no gravitational radiation since the metric exterior to the distribution is static [3] and [93]. Eq. (2.40) is usually referred to as the exterior solution. For a stellar model one may similarly obtain the interior Schwarzschild solution by making use of the equations of state for a star [3]. We just state the metric for the interior solution, namely

$$d\hat{s}^2 = -\left[\frac{3}{2}\sqrt{1-\frac{r_0^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1-\frac{r^2}{\hat{R}^2}}\right]^2 dt^2 + \left(1-\frac{r^2}{\hat{R}^2}\right)^{-1} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (2.41)$$

for $r \leq r_0$, and $\hat{R}^2 = \frac{3}{8\pi G\rho}$, where r_0 is the radius of the model star, and $\hat{R}^2$ is a quantity with the dimensions of a length, $\hat{\rho}$ is the constant density of the star and G the gravitational constant. Birkhoff's theorem may also be applied inside an empty spherical cavity at the center of a spherically symmetric body. This is stated in the following corollary of Birkhoff's theorem: *The metric inside an empty spherically symmetric cavity at the centre of a spherically symmetric star must be equivalent to the flat-space Minkowski metric η_{ij}.*

2.5 Schwarzschild geometry

In this section we shall look at the Schwarzschild geometry with the view of finding the complete analytic extension of this geometry from the exterior into the interior of the Schwarzschild black hole by way of coordinate transformation. This shall help us to understand the causal structure of the Schwarzschild geometry.

For the sake of definiteness, from Eq. (2.40) we shall consider only the case $M > 0$. We now make

$$\upsilon = t + r + 2M\log(r - 2M), \quad (2.42)$$

where υ is constant on radially ingoing null geodesics, giving the ingoing extension

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\upsilon^2 + 2d\upsilon dr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.43}$$

Now the Schwarzschild geometry is time-symmetric (invariant under $t \to -t$), and so it may also be rewritten using the null coordinate

$$u = t - r - 2M\log(r - 2M), \tag{2.44}$$

which is constant on radially outgoing null geodesics. This gives the out going extension

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.45}$$

By combining together the ingoing and outgoing extensions, we get the double-null form given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dud\upsilon + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.46}$$

Here r is given implicitly as a function of u and υ through

$$\frac{1}{2}(\upsilon - u) = r + 2M\log(r - 2M). \tag{2.47}$$

The logarithm implies that $r(u, \upsilon)$ is badly behaved at $r = 2M$.

Now we redefine the null coordinates [53]:

$$\hat{u} = -\exp(-u/4M), \tag{2.48}$$

$$\hat{\upsilon} = \exp(\upsilon/4M). \tag{2.49}$$

The metric then takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) \frac{d\upsilon}{d\hat{\upsilon}} \frac{du}{d\hat{u}} d\hat{u} d\hat{\upsilon} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.50}$$

We now define

$$\hat{r} = \frac{1}{2}(\hat{\upsilon} - \hat{u}), \quad \hat{t} = \frac{1}{2}(\hat{\upsilon} + \hat{u}). \tag{2.51}$$

The metric then becomes

$$ds^2 = -F^2(\hat{t}, \hat{r})(-d\hat{t}^2 + d\hat{r}^2) + r^2(\hat{t}, \hat{r})(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.52}$$

where $r(\hat{t},\hat{r})$ is given implicitly by

$$(\hat{t})^2 - (\hat{r})^2 = -(r - 2M)\exp(r/2M), \tag{2.53}$$

and

$$F^2(\hat{t},\hat{r}) = \frac{16M^2}{r}\exp\left(\frac{-r}{2M}\right), \tag{2.54}$$

Eq. (2.53) is now invertible for $0 < r < \infty$ and the curvature singularity at r=0 implies that one should only consider values of r with $r > 0$.

Now

$$\frac{\hat{t}}{\hat{r}} = \tanh\left(\frac{t}{4M}\right). \tag{2.55}$$

and the domain of validity of the form (2.52) of the Schwarzschild geometry is

$$\{\hat{t},\hat{r} : r(\hat{t},\hat{r}) > 0\} = \{\hat{t},\hat{r} : (\hat{t})^2 - (\hat{r})^2 < 2M\}. \tag{2.56}$$

Then one arrives at the Kruskal diagram of the maximally extended Schwarzschild geometry depicted in fig. 2.1 below

Figure 2.1: The Kruskal diagram of the Schwarzschild geometry: Light cones are at 45 degrees. The future event horizon can again be identified as a surface with $r = 2M$. There are two distinct singularities at r=0, one in the past and one in the future.

We note from the figure that the light cones are at 45 degrees to the vertical axis. We also note that there are two null surfaces with $r = 2M$, which meet at a point in the center of the figure representing a two-sphere. Also there are two singular 'surfaces' with $r = 0$. One cannot regard a singular region as part of the space time manifold, but it should be evident from fig. 2.1 that the two singularities are in some sense space-like. Figure 2.2 below shows the maximally extended Schwarzschild geometry depicting the intrinsic spatial geometry of a section $t = \text{const}$

Figure 2.2: The intrinsic spatial geometry of a section $t = \text{const}$: $\theta = \pi/2$ through the maximally extended Schwarzschild geometry, visualized by embedding in flat Euclidean three-space.

2.6 Carter-Penrose diagrams and causal properties

We construct a Carter-Penrose conformal diagram for the Schwarzschild geometry by changing variables from the metric in the Kruskal coordinates. We define [45, 66]

$$\hat{\hat{u}} = \tan^{-1}\left(\frac{\hat{u}}{\sqrt{(2M)}}\right), \quad \hat{\hat{v}} = \tan^{-1}\left(\frac{\hat{v}}{\sqrt{(2M)}}\right). \tag{2.57}$$

This gives the maximally extended vacuum Schwarzschild solution with null infinity brought into finite coordinates. The domain is then given by

$$-\frac{\pi}{2} < \hat{\hat{u}} + \hat{\hat{v}} < \frac{\pi}{2}; \quad -\frac{\pi}{2} < \hat{\hat{u}} < \frac{\pi}{2}; \quad -\frac{\pi}{2} < \hat{\hat{v}} < \frac{\pi}{2}. \tag{2.58}$$

The resulting Carter-Penrose conformal diagram is given in fig. 2.3 below which shows, in addition lines of constant t and r respectively.

Figure 2.3: The Carter-Penrose diagram of the maximally extended Schwarzschild geometry: In (a), lines of constant t are shown and in (b), lines of constant r are shown. The future event horizon is the surface marked S.

Each point in the Carter-Penrose diagram represents a two-sphere S^2. We can also verify from this construction that the Schwarzschild space-time is weakly asymptotically simple, having two sets of asymptotic regions, either $\mathcal{I}^-$, $\mathcal{I}^+$ as on the right hand of fig. 2.3 (a), or $\mathcal{I}^-$, $\mathcal{I}^+$ on the left hand side of fig. 2.3 (a). We define the concept of asymptotically simple in the following theorem [88]:

Theorem 2.6.1 *A space-time* $(\mathcal{M}, g)$ *is asymptotically simple if* $\exists$ *a manifold* $(\widetilde{\mathcal{M}}, \widetilde{g})$ *with boundary* $\partial\widetilde{\mathcal{M}} = \overline{\mathcal{M}}$ *and a continuous embedding* $f(\mathcal{M}) : \mathcal{M} \to \widetilde{\mathcal{M}}$ *s.t.*

1. $f(\mathcal{M}) = \widetilde{\mathcal{M}} - \partial\widetilde{\mathcal{M}}$

2. $\exists$ a smooth function $\mathcal{F}$ on $\widetilde{\mathcal{M}}$ with $\mathcal{F} > 0$ on $f(\mathcal{M})$ and $\widetilde{g} = \mathcal{F}^2 f(g)$.

3. $\mathcal{F} = 0$ but $d\mathcal{F} \neq 0$ on $\partial\widetilde{\mathcal{M}}$.

4. Every null geodesic in $\mathcal{M}$ acquires 2 end points on $\partial\mathcal{M}$.

where $\widetilde{\mathcal{M}}$ is the conformal 'compactification' of $\mathcal{M}$, and $\widetilde{g}$ is the conformal 'compactification' of g.

Also from fig. 2.3. one can read immediately that the Schwarzschild space-time contains a black hole, namely the region above the null surface S with $r = 2M$ and we also note that this region cannot send signals to $\mathcal{I}^+$. The singularities $r = 0$ are space-like [60, 88]. The final singularity is hidden from view from $\mathcal{I}^+$ by the future event horizon[2] S. But the initial singularity at $r = 0$, is visible from $\mathcal{I}^+$ so that the extended vacuum Schwarzschild geometry has a naked singularity[3] [19].

The domain for the maximally extended Schwarzschild geometry can be divided into four regions (i.e. I, II, II, IV) depicted in fig. 2.4 below

[2]By the future event horizon we mean: Assuming that the space-time $\mathcal{M}$ is weakly asymptotically flat, then defining $\mathcal{K}^-(\mathcal{U})$ to be the *causal past* of a set of points $\mathcal{U} \subset S$ and $\overline{\mathcal{K}}^-(\mathcal{U})$ to be the topological closure of $\mathcal{K}^-$, i.e. including limits points. Then defining again the boundary of $\overline{\mathcal{K}}^-$ to be $\dot{\mathcal{K}}^-(\mathcal{U}) = \overline{\mathcal{K}}^-(\mathcal{U}) - \mathcal{K}^-(\mathcal{U})$. Then the *future even horizon* of $\mathcal{M}$ is the null hypersurface $S = \dot{\mathcal{K}}^-(\mathcal{I}^+)$, i.e. the *boundary of the closure of the causal of* $\mathcal{I}^+$ [88]

[3]The nakedness of the singularity offends our sense of decency, as well as the **cosmic censorship conjecture**, which roughly states that the gravitational collapse of physical matter configurations will never produce a naked singularity. This conjecture may not be right, though there are some claims from numerical simulations that the collapse of spindle like configurations can lead to naked singularities [19].

Figure 2.4: Four Carter-Penrose diagrams showing the different regions in the Schwarzschild geometry covered by different coordinates: (a) Schwarzschild coordinates; (b) (v,r) Eddington-Finkelstein coordinates; (c) (u,r) Eddington-Fenkelstein coordinates; (d) Kruskal coordinates.

Figure 2.4 shows how different groupings of these regions are obtained from different coordinate systems which have been used. Region I is a normal space-time outside the black hole and is accessible from $\mathcal{I}^-$ by physical objects with speed < c, where c speed of light. Region II is a space-time inside the even horizon and is accessible from $\mathcal{I}^-$ by physical objects with speed < c. Regions III and IV are just like II and I respectively and they arise from the maximal extension of the Schwarzschild solution (2.40). Region III acts like a black hole in reverse(a white hole) and matter can be ejected out of the past singularity in this region. If a black hole is formed by a collapsing matter, regions III and IV are shielded from I and II and in this case we don't have to consider III and IV as physical. Mathematically speaking, a black hole can exist independently of a collapsing matter and if there is no shield due to collapsing matter, then I and IV are connected by a *wormhole*.

2.7 Description of the Schwarzschild geometry in the Newman-Penrose formalism

A basic null-tetrad satisfying the requirements

$$l^i \cdot n^i = 1, \quad m^i \cdot \overline{m^i} = -1, \quad \kappa = \varpi = \lambda = \varsigma = 0, \tag{2.59}$$

and appropriate to the Schwarzschild metric is given by the following contravariant basis vectors:

$$l^i = (l^0, l^1, l^2, l^3) = \frac{1}{\Delta}(r^2, +\Delta, 0, 0), \tag{2.60}$$

$$n^i = (n^0, n^1, n^2, n^3) = \frac{1}{2r^2}(r^2, -\Delta, 0, 0), \tag{2.61}$$

where

$$\Delta = r^2 - 2Mr, \tag{2.62}$$

and the complex basis null-vector by

$$m^i = (m^0, m^1, m^2, m^3) = \frac{1}{r\sqrt{2}}(0, 0, 1, i\mathrm{cosec}\theta), \tag{2.63}$$

and the corresponding covariant basis vectors l^i, n^i, m^i, and $\overline{m}^i$ by

$$
\begin{aligned}
l_i &= (1, -\frac{r^2}{\Delta}, 0, 0), &(2.64)\\
n_i &= \frac{1}{2r^2}(\Delta, +r^2, 0, 0), &(2.65)\\
m_i &= \frac{1}{r\sqrt{2}}(0, 0, -r^2, -ir^2 \sin\theta). &(2.66)
\end{aligned}
$$

The remaining spin coefficients (as defined in chapter 1.4, Eq. (1.27)) determined by this basis vectors are given by:

$$\rho = -\frac{1}{r}, \qquad -\alpha = \vartheta = \frac{\cot\theta}{r(2\sqrt{2})}, \qquad \pi = \tau = \varnothing = 0, \tag{2.67}$$

$$\wp = -\frac{\Delta}{2r^3}, \quad \text{and} \quad \gamma = \wp + \frac{r-M}{2r^2} = \frac{M}{2r^2}. \tag{2.68}$$

It can also be verified that, with respect to the chosen basis,

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \tag{2.69}$$

and

$$\Psi_2 = -Mr^{-3}. \tag{2.70}$$

This completes the description of the Schwarzschild geometry in the Newman-Penrose formalism and the results of this section are needed in section 3.5 where we shall be discussing the Bardeen-Press equation.

2.8 Conclusion

We introduced the concept of spherically symmetric space-times, derived Schwarzschild solution and stated Birkhorff's theorem. We also looked at Schwarzschild geometry and its Carter-Penrose diagrams and causal properties, and lastly we described Schwarzschild geometry in Newman-Penrose formalism. In the next chapter, we shall perturb a Schwarzschild black hole by linearly perturbing Schwarzschild solution introduced in this chapter but in a more general setup described by Chandrasekhar ([25] see chap. 1, sec. 12) when he is talking about generalizing Schwarzschild solution to include situations which are non-stationary and non-axisymetric.

Chapter 3

Linear perturbations of a Schwarzschild black hole

3.1 Introduction

In this chapter we introduce Ricci tensors for non-stationary axisymmetric space-times and derive the Zerilli and Regge-Wheeler equations and show that they are related via their potentials and their one-dimensional wave functions. We also derive the Bardeen-Press equation using the Newman-Penrose formalism and show that it is related to both the Zerilli and Regge-Wheeler equations. In deriving and showing the connection between these equations, we shall use the method used by Chandrasekhar [24, 25]. Lastly we introduce the concepts of quasi-normal modes of a Schwarzschild black hole with reference to the work of Leaver [55] and Chandrasekhar [24, 25] with the aim of using them later in chapter 5.

3.2 The Ricci and the Einstein tensors for non-stationary axisymmetric space-times

We restrict ourself to axisymmetric modes without any loss of generality. This is because in a case of non-axisymmetric modes with an $e^{im\phi}$ dependence on the azimuthal angle ϕ (where m is a non-zero integer), we can, by suitable rotations deduce them from the axisymmetric modes with $m = 0$ because there are no preferred axes in a spherically symmetric background. When an axisymmteric mode is expressed in another frame with its polar axis pointing in a direction $(\grave{\theta}, \grave{\phi})$ evaluated at a point (θ, ϕ) on the sphere with respect to a chosen polar axis we assign it a polar angle Θ given by

$$\cos\Theta = \cos\theta\cos\grave{\theta} + \sin\theta\sin\grave{\theta}\cos(\grave{\phi} - \phi). \tag{3.1}$$

Now Eq. (3.1) will cause an axisymmetric mode to decompose into different non-axisymmetric modes. Due to the above analysis, we restrict ourselves to time-dependent axisymmetric modes for linear perturbations of a Schwarzschild black hole. We now consider a Schwarzschild solution to be a special solution (spherically symmetric time-independence) of the field equations appropriate to the line element [20]

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\lambda}(d\phi - \omega dt - q_2 dx^2 - q_3 dx^3)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2, \tag{3.2}$$

where

$$e^{2\nu} = e^{-2\mu_2} = 1 - 2M/r = \Delta/r^2, \quad e^{\mu_3} = r, \quad e^{\lambda} = r\sin\theta, \tag{3.3}$$

and

$$\omega = q_2 = q_3 = 0 \quad (x^0 = t,\ x^1 = \phi,\ x^2 = r,\ x^3 = \theta). \tag{3.4}$$

In the next section, we shall see that linear perturbation equations (Zerilli and Regge-Wheeler) are obtained by linearizing the field equations about the Schwarzschild solution. We have computationally recalculated the Ricci tensors and linearized them about Schwarzschild and checked them against the analytic [1] results originally calculated by Chandrasekhar [20, 24]. From our calculations (see Appendix A for a Maple program and the results), we found that our results differ from that of Chandrasekhar by some common factors. In the table below we show the factors that did not appear in R_{00}, R_{11}, R_{22}, R_{33}, R_{01}, R_{02}, R_{13}, R_{12}, R_{23}, where factor is defined as

$$\text{factor} = \frac{\text{linearized value of Ricci tensor reported by Chandrasekhar}}{\text{linearized value of Ricci tensor found by Maple program}}, \tag{3.5}$$

Ricci tensor	Factors
R_{00}	$e^{-2\nu}$
R_{11}	$-e^{-2\lambda}$
R_{22}	$e^{-2\mu_2}$
R_{33}	$-e^{-2\mu_3}$
R_{01}	$e^{-\nu-\lambda}$
R_{02}	$-e^{-\mu_2-\nu}$
R_{13}	$e^{-\lambda-\mu_3}$
R_{12}	$e^{-\mu_2-\lambda}$
R_{23}	$-e^{-\mu_3-\mu_2}$

Table 3.1: Table for the missing factors for the Ricci tensor calculated by Chandrasekhar.

In the non linear regime the error in Chandrasekhar's results is more than just a common factor. Below we give as an example, the non-linear terms that did not appear in R_{00} (See Appendix A.3 for a Maple program and the results in A.4, note R00 in the program is R44).

$$\begin{aligned} ChR00 - (checkR00)(Ricci[0,0]) = 1/2\omega \big[e^{-4\nu+2\lambda} \{2\omega(\lambda_{,0,0} + \lambda_{,0}(\lambda+\mu_3+\mu_2-\nu)_{,0})\} \\ +e^{-2\nu-2\mu_2+2\lambda} \{-2\omega(\lambda_{,2,2}+\lambda_{,2}(\lambda+\mu_3-\mu_2+\nu)_{,2}) + 2Q_{20}(\mu_3-\mu_2-\nu+3\lambda)_{,2} + 2Q_{20,2}\} \\ +e^{-2\nu-2\mu_3+2\lambda} \{-2\omega(\lambda_{,3,3}+\lambda_{,3}(\lambda+\mu_2+\nu-\mu_3)_{,3}) + 2Q_{30}(-\mu_3+\mu_2+3\lambda-\nu)_{,3} + 2Q_{30,3}\} \\ -\omega Q_{20}^2 e^{-4\nu+4\lambda-2\mu_2} - \omega Q_{30}^2 e^{-4\nu-2\mu_3+4\lambda} + \omega Q_{23}^2 e^{-2\nu-2\mu_2+4\lambda-2\mu_3} \big] . \end{aligned} \tag{3.6}$$

where $Q_{AB} = q_{A,B} - q_{B,A}$, $Q_{A0} = q_{A,0} - \omega_{,A}$, $(A, B = 2, 3)$ and $ChR00$ refers to Chandrasekhar's original version, $Ricci[0,0]$ is our computationally calculated version, and $checkR00$ is the computed factor for R_{00} shown in the table above.

The discrepancies reported above do not invalidate any of the results derived by Chandrasekhar concerning linear perturbations of a Schwarzschild black hole, because

[1] Chandrasekhar used Cartan's exterior calculus to calculate analytically these Ricci tensors.

the imposition of the two conditions (i) linearization and (ii) in vacuum means that the Einstein equations are not affected.

In the next section we shall see that in the linearized regime the constant factors mentioned above for R_{12} and R_{13} does not have any affect on the linear perturbations of a Schwarzschild black hole.

3.3 The Regge-Wheeler and Zerilli equations

The Zerilli and Regge-Wheeler equations each describe one of the two degrees of freedom of linearized gravity propagating in a black hole background [52, 67]. From Eq. (3.2), linear perturbations[2] of a Schwarzschild black hole will result in ω, q_2, and q_3 becoming small quantities of the first order resulting in odd-parity perturbations which are governed by Regge-Wheeler equation, and the functions ν, μ_2, μ_3, and λ experiencing small increments $\delta\nu$, $\delta\mu_2$, $\delta\mu_3$, and $\delta\lambda$ resulting in even-parity perturbation which are governed by the Zerilli equation. Odd-parity perturbations in the literature are often referred to as *axial perturbations* because they cause the inertial frame to be dragged and thus resulting in a black hole rotating. Also in the literature, even-parity perturbations are referred to as *polar perturbations* and they do not cause the black hole to rotate.

3.3.1 The Regge-Wheeler equation

We start by noting that the equations governing ω, q_2, and q_3, are given by

$$R_{12} = R_{13} = 0, \tag{3.7}$$

By inserting the unperturbed values of ν, μ_2, μ_3, and λ stated in Eq. (3.3) into Eq. (3.7) we get

$$(e^{3\lambda+\nu-\mu_2-\mu_3} Q_{23})_{,3} \quad = \quad -e^{3\lambda-\nu+\mu_3-\mu_2} Q_{02,0} \quad (\delta R_{12} = 0), \tag{3.8}$$

[2] Chandrasekhar used a different notation for linear perturbations of a Schwarzschild black hole, instead of using h_{ij} that is used by other authors in the literature, he used ω, q_2, q_3, $\delta\nu$, $\delta\mu_2$, $\delta\mu_3$, and $\delta\lambda$.

$$(e^{3\lambda+\nu-\mu_2-\mu_3}Q_{23})_{,2} \quad = \quad +e^{3\lambda-\nu+\mu_2-\mu_3}Q_{03,0} \quad (\delta R_{13}=0). \tag{3.9}$$

Letting

$$Q(t,r,\theta) = \Delta Q_{23}\sin^3\theta = \Delta(q_{2,3}-q_{3,2})\sin^3\theta, \tag{3.10}$$

and again substituting the unperturbed values of ν, $\wp_2$, $\wp_3$, and λ in Eq. (3.7), we obtain the pair of equations

$$\frac{1}{r^4\sin^3\theta}\frac{\partial Q}{\partial\theta} = -(\omega_{,2}-q_{2,0})_{,0}, \tag{3.11}$$

$$\frac{\Delta}{r^4\sin^3\theta}\frac{\partial Q}{\partial r} = (\omega_{,3}-q_{3,0})_{,0}. \tag{3.12}$$

By assuming that the perturbations have the time-dependency factor given by $e^{i\sigma t}$ where σ is a constant which is real in general and retaining the same symbols for the amplitudes of the perturbations with the time-dependency factor, Eqs. (3.11) and (3.12) becomes

$$\frac{1}{r^4\sin^3\theta}\frac{\partial Q}{\partial\theta} = -i\sigma\omega_{,2}-\sigma^2 q_2, \tag{3.13}$$

and

$$\frac{\Delta}{r^4\sin^3\theta}\frac{\partial Q}{\partial r} = +i\sigma\omega_{,3}+\sigma^2 q_3. \tag{3.14}$$

After eliminating ω from Eqs. (3.13) and (3.14), we get

$$r^4\frac{\partial}{\partial r}\left(\frac{\Delta}{r^4}\frac{\partial Q}{\partial r}\right)+\sin^3\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^3\theta}\frac{\partial Q}{\partial\theta}\right)+\sigma^2\frac{r^4}{\Delta}Q=0. \tag{3.15}$$

From here, we are now able to separate the variables r and θ in Eq. (3.15) by making the substitution

$$Q(r,\theta) = Q(r)C_{l+2}^{-3/2}(\theta), \tag{3.16}$$

where C_n^ς denotes the Gegenbauer function given by

$$\left[\frac{d}{d\theta}\sin^{2\varsigma}\theta\frac{d}{d\theta}+n(n+2\varsigma)\sin^{2\varsigma}\theta\right]C_n^\varsigma = 0, \tag{3.17}$$

the Gegenbauer function $C_{l+2}^{-3/2}(\theta)$, is related to the Legendre function $P_l(\theta)$ by

$$C_{l+2}^{-3/2}(\theta) = \sin^3\theta\frac{d}{d\theta}\frac{1}{\sin\theta}\frac{dP_l(\theta)}{d\theta}, \tag{3.18}$$

or

$$C_{l+2}^{-3/2}(\theta) = (P_{l,3,3}-P_{l,3}\cot\theta)\sin^2\theta. \tag{3.19}$$

After substituting Eq. (3.16) in Eq. (3.15), we get the radial equation

$$\Delta \frac{d}{dr}\left(\frac{\Delta}{r^4}\frac{dQ}{dr}\right) - \epsilon^2 \frac{\Delta}{r^4}Q + \sigma^2 Q = 0, \tag{3.20}$$

where

$$\epsilon^2 = 2n = (l-1)(l+2) \tag{3.21}$$

specifies the associated angular dependence. Then lastly by changing to the tortoise coordinate $\hat{r}_*$ given by

$$\hat{r}_* = r + 2M \log\left(\frac{r}{2M} - 1\right) \quad \left(\frac{d}{d\hat{r}_*} = \frac{\Delta}{r^2}\frac{d}{dr}\right), \tag{3.22}$$

and letting

$$Q(r) = rZ^{(-)}, \tag{3.23}$$

we find that $Z^{(-)}$ satisfies the one-dimensional Schrodinger wave equation given by

$$\left(\frac{d^2}{d\hat{r}_*^2} + \sigma^2\right) Z^{(-)} = V^{(-)} Z^{(-)}, \tag{3.24}$$

where the potential $V^{(-)}$ is given by

$$V^{(-)} = \frac{\Delta}{r^5}[(\epsilon^2 + 2)r - 6M]. \tag{3.25}$$

Eq. (3.24) is called *Regge-Wheeler equation* and was derived by Regge and Wheeler [70] in 1957.

3.3.2 The Zerilli equation

We start by linearizing R_{02}, R_{03}, R_{23}, and R_{11} (see Appendix B, for a Maple program that calculates these Ricci tensors) about Schwarzschild values and we get

$$(\delta\lambda + \delta\mu_3)_{,2} + (1/r - \varsigma_{,2})(\delta\lambda + \delta\mu_3) - (2/r)\delta\mu_2 \quad (\delta R_{02} = 0). \tag{3.26}$$

$$(\delta\psi + \delta\mu_2)_{,3} + (\delta\lambda - \delta\mu_3)\cot\theta = 0 \quad (\delta R_{03} = 0), \tag{3.27}$$

$$(\delta\lambda + \delta\nu)_{,2,3} + (\delta\lambda - \delta\mu_3)_{,2}\cot\theta + \left(\nu_{,2} - \frac{1}{r}\right)\delta\nu_{,3} - \left(\nu_r + \frac{1}{r}\right)\delta\mu_{2,3} = 0 \quad (\delta R_{23} = 0), \tag{3.28}$$

and

$$e^{-2\mu_2}\left[\frac{2}{r}\delta\nu_{,2}+\left(\frac{1}{r}+\nu_{,2}\right)(\delta\psi+\delta\mu_3)_{,2}-2\delta\mu_2\left(\frac{1}{r^2}+2\frac{\nu_{,2}}{r}\right)\right]$$
$$+\frac{1}{r^2}[(\delta\lambda+\delta\nu)_{,3,3}+(2\delta\lambda+\delta\varsigma-\delta\mu_3)_{,3}\cot\theta+2\delta\mu_3]$$
$$-e^{-2\nu}(\delta\lambda+\delta\mu_3)_{,0,0}=0\quad(\delta G_{22}=0). \tag{3.29}$$

We find the following equation for $\delta R_{11}=0$ to be useful [21, 25]:

$$\begin{aligned} e^{+2\nu} &\left[\delta\lambda_{,2,2}+2\left(\frac{1}{r}+\nu_{,2}\right)\delta\lambda_{,2}+\frac{1}{r}(\delta\lambda+\delta\nu+\delta\mu_3-\delta\mu_2)_{,2}-2\frac{\delta\mu_2}{r}\left(\frac{1}{r}+2\nu_{,2}\right)\right] \\ + &\frac{1}{r^2}[\delta\lambda_{,3,3}+\delta\lambda_{,3}\cot\theta+(\delta\lambda+\delta\nu-\delta\mu_3+\delta\mu_2)_{,3}\cot\theta+2\delta\mu_3] \\ - &e^{-2\nu}\delta\lambda_{,0,0}=0. \end{aligned} \tag{3.30}$$

We now separate the variables r and θ in Eqs. (3.26), (3.27), (3.28), (3.29) and (3.30) by making the following substitutions

$$\begin{aligned} \delta\nu &= N(r)P_l(\cos\theta)e^{i\sigma t}, \\ \delta\mu_2 &= \delta\mu_2=L(r)P_l(\cos\theta)e^{i\sigma t}, \\ \delta\mu_3 &= [T(r)P_l+V(r)P_{l,3,3}]e^{i\sigma t}, \\ \text{and}\quad \delta\lambda &= [T(r)P_l+V(r)P_{l,3}\cot\theta]e^{i\sigma t}, \end{aligned} \tag{3.31}$$

where N, L, T, and V are radial functions and we have assumed a time dependency factor $e^{i\sigma t}$.

From Eq. (3.27), with the above substitutions we get

$$T-V+L=0. \tag{3.32}$$

Eq. (3.32) implies that N, T and V are linearly independent and we choose N, L, and V as the independent functions.

Also, by the above substitutions the components of R_{02} and R_{23} of the field equations, Eqs. (3.26) and (3.28), give

$$\left[\frac{d}{dr}+\left(\frac{1}{r}-\nu_{,2}\right)\right][2T-l(l+1)V]-\frac{2}{r}L=0. \tag{3.33}$$

and

$$(T - V + N)_{,2} - \left(\frac{1}{r} - \nu_{,2}\right) N - \left(\frac{1}{r} + \nu_{,2}\right) L = 0. \tag{3.34}$$

Then by eliminating T with the aid of Eq. (3.32), we obtain

$$N_{,2} - L_{,2} = \left(\frac{1}{r} - \nu_{,2}\right) N + \left(\frac{1}{r} + \nu_{,2}\right) L \tag{3.35}$$

and

$$L_{,2} + \left(\frac{2}{r} - \nu_{,2}\right) L = -n \left[V_{,2} + \left(\frac{1}{r} - \nu_{,2}\right) V\right]. \tag{3.36}$$

Similarly, Eqs. (3.28) and (3.29) give

$$\begin{aligned} &\frac{2}{r} N_{,2} + \left(\frac{1}{r} + \nu_{,2}\right) [2T - l(l+1)V]_{,2} - \frac{2}{r}\left(\frac{1}{r} + 2\nu_{,2}\right) L \\ &-l(l+1)\frac{e^{-2\nu}}{r^2} N - 2n\frac{e^{-2\nu}}{r^2} T + \sigma^2 e^{-4\nu}[2T - l(l+1)V] = 0, \end{aligned} \tag{3.37}$$

and

$$V_{,2,2} + 2\left(\frac{1}{r} + \nu_{,2}\right) V_{,2} + \frac{e^{-2\nu}}{r^2}(N + L) + \sigma^2 e^{-4\nu} V = 0. \tag{3.38}$$

Eq. (3.37) after the elimination of T takes the form

$$\begin{aligned} &\frac{2}{r} N_{,2} - l(l+1)\frac{e^{-2\nu}}{r^2} N - \frac{2}{r}\left(\frac{1}{r} + 2\nu_{,2}\right) L - 2\left(\frac{1}{r} + \nu_{,2}\right)(L + nV)_{,2} \\ &\qquad -2n\frac{e^{-2\nu}}{r^2}(V - L) - 2\sigma^2 e^{-4\nu}(L + nV) = 0, \end{aligned} \tag{3.39}$$

where

$$n = \frac{1}{2}(l-1)(l+2). \tag{3.40}$$

Eqs. (3.35), (3.36) and (3.39) provide three linear first order equations for the three radial functions L, N, and V. By combining these equations we are able to express the first derivative of each of them as linear combinations of L, N, and V, thus, we have

$$N_{,2} = c_1 N + c_2 L + c_3 X, \tag{3.41}$$

$$L_{,2} = \left(c_1 - \frac{1}{r} + \nu_{,2}\right) N + \left(c_2 - \frac{1}{r} - \nu_{,2}\right) L + c_3 X \tag{3.42}$$

and

$$X_{,2} = -\left(c_1 - \frac{1}{r} + \nu_{,2}\right) N - \left(c_2 + \frac{1}{r} - 2\nu_{,2}\right) L - \left(c_3 + \frac{1}{r} - \nu_{,2}\right) X, \tag{3.43}$$

where

$$X = nV = \frac{1}{2}(l-1)(l+2)V, \tag{3.44}$$

$$c_1 = \frac{n+1}{r-2M}, \qquad \nu_{,2} = \frac{M}{r(r-2M)},$$
$$c_2 = -\frac{1}{r} - \frac{n}{r-2M} + \frac{M}{r(r-2M)} + \frac{M^2}{r(r-M)^2} + \sigma^2 \frac{r^3}{(r-2M)^2}$$
$$c_3 = -\frac{1}{r} + \frac{1}{r-2M} + \frac{M^2}{r(r-2M)^2} + \sigma^2 \frac{r^3}{(r-2M)^2}. \tag{3.45}$$

Eqs. (3.41), (3.42) and (3.43) are the basic equations that we shall work with and all the remaining field equations, including Eq. (3.38), are verifiable consequences of these equations.

By adding Eqs. (3.42) and (3.43) we obtain

$$(L+X)_{,2} = -\left(\frac{2}{r} - \nu_{,2}\right) L - \left(\frac{1}{r} - \nu_{,2}\right) X, \tag{3.46}$$

Then finally, we reduce Eqs. (3.41), (3.42) and (3.43) to a one-dimensional wave equation as follows:

Given the function

$$Z^{(+)} = \frac{r^2}{nr+3M}\left(\frac{3M}{r}V - L\right), \tag{3.47}$$

and by virtue of Eqs. (3.41), (3.42), and (3.43), this function satisfies a one dimensional wave equation [21].

Rewriting Eq. (3.47) to its equivalent form we get

$$Z^{(+)} = rV - \frac{r^2}{nr-3M}(L+X). \tag{3.48}$$

Differentiating Eq. (3.48) with respect to the variable

$$\hat{r}_* = r + 2M \log\left(\frac{r}{2M} - 1\right) \qquad (r > 2M), \tag{3.49}$$

and making use of Eq. (3.46), we obtain

$$Z^{(+)}_{\hat{r}_*} = \left(1 - \frac{2M}{r}\right) Z_{,2} \tag{3.50}$$
$$= (r-2M)V_{,2} + \frac{3M(r-2M)}{r(nr+3M)}V$$
$$+ \frac{nr^2 - 3nMr - 3M^2}{(nr+3M)^2}(L+X). \tag{3.51}$$

Differentiating Eq. (3.51) once again with respect to $\hat{r}_*$ and making use of Eq. (3.46) once again and simplifying with the aid of Eqs. (3.38), (3.43) and (3.48) we find after some considerable reductions that,

$$\frac{d^2 Z^{(+)}}{d\hat{r}_*^2} + (\sigma^2 - V_z^{(+)}) Z^{(+)} = 0 \quad (+\infty > \hat{r}_* > -\infty), \tag{3.52}$$

where

$$V_z^{(+)} = \frac{2n^2(n+1)r^3 + 6n^2 M r^2 + 18 n M^2 r + 18 M^3}{r^3 (nr + 3M)^2} \left(1 - \frac{2M}{r}\right), \tag{3.53}$$

$$\frac{d}{d\hat{r}_*} = \left(1 - \frac{2M}{r}\right) \frac{d}{dr} \quad \text{and} \quad n = \frac{1}{2}(l-1)(l+2). \tag{3.54}$$

Eq. (3.52) is called the *Zerilli equation* [99] and was first derived by him in (1970). Eqs. (3.24) and (3.52) have the properties that they reduce to the Klein-Gordon wave equation in the flat-space limit [52].

3.4 The relation between Regge-Wheeler and Zerilli equations

In this section we are going to show the relation between Regge-Wheeler and Zerilli equations by firstly showing the relation between their potentials $V^{(-)}$ and $V^{(+)}$ respectively and secondly by showing the relation between their one-dimensional Schrodinger wave functions $Z^{(-)}$ and $Z^{(+)}$ respectively.

3.4.1 The relation between $V^{(-)}$ and $V^{(+)}$

The two potentials $V^{(-)}$, $V^{(+)}$ given by Eq. (3.25) and Eq. (3.53) respectively, turns out to be simply related in such a way that they satisfy the following relation [25]:

$$V^{(\pm)} = \pm \mathfrak{A} \frac{df}{d\hat{r}_*} + \mathfrak{A}^2 f^2 + \mathfrak{B} f, \tag{3.55}$$

where

$$\mathfrak{A} = \text{constant} = 6M, \quad \mathfrak{B} = \text{constant} = 4n(n+1) = \epsilon^2(\epsilon^2 + 2), \tag{3.56}$$

and

$$f = \frac{\Delta}{r^3(\epsilon^2 r + 6M)} = \frac{\Delta}{2r^3(nr + 3M)}. \tag{3.57}$$

The function f vanishes both at the horizon ($r = 2M$) and at infinity with an inverse-square r^{-2} behavior.

The origin of the relation (3.55) shall emerge in section (3.5) were we shall be using the Newman-Penrose formalism to derive the Bardeen-Press equation that also describes linear perturbations of a Schwarzschild black hole. In the meantime, we shall accept this relation to be a direct verifiable fact and in the next section, we shall show that it implies a relation between $Z^{(-)}$ and $Z^{(+)}$.

3.4.2 The relation between $Z^{(-)}$ and $Z^{(+)}$

For the convenience of notation, We shall temporarily replace $\hat{r}_*$ by x and $Z^{(+)}$ and $Z^{(-)}$ by Z_1 and Z_2 respectively.

We start with the following two wave equations

$$\frac{d^2 Z_1}{dx^2} + \sigma^2 Z_1 = V_1 Z_1 = \left(+\mathfrak{A}\frac{df}{dx} + \mathfrak{A}^2 f^2 + \mathfrak{B} f \right) Z_1 \tag{3.58}$$

and

$$\frac{d^2 Z_2}{dx^2} + \sigma^2 Z_2 = V_2 Z_2 = \left(-\mathfrak{A}\frac{df}{dx} + \mathfrak{A}^2 f^2 + \mathfrak{B} f \right) Z_2, \tag{3.59}$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are real constants and f is an arbitrary smooth function and together with its derivatives of all orders they vanish for both $x \to +\infty$ and $x \to -\infty$. The integral of f over the entire range of x is considered to be finite.

Given a solution of Z_2 of equation (3.59), and suppose that

$$Z_1 = pZ_2 + qZ_2' \tag{3.60}$$

is a solution of Eq. (3.58), where Z_2' denotes the first derivative of Z_2 with respect to x, and p, q are certain suitably chosen functions. We now derive the equations that must govern p and q in order that Z_1 given by Eq. (3.60) is a solution of Eq. (3.58).

Firstly we start by differentiating Eq. (3.60) and making use of Eq. (3.59) satisfied by Z_2 to get

$$Z_1' = [p' + q(V_2 - \sigma^2)]Z_2 + (p + q')Z_2'. \tag{3.61}$$

Then secondly we differentiate Eq. (3.61) once again, to get

$$Z_1'' = [p'' + (p + 2q')(V_2 - \sigma^2) + qV_2']Z_2 + [p' + q(V_2 - \sigma^2) + p' + q'']Z_2'. \tag{3.62}$$

We note that Eq. (3.62) is identical to

$$Z_1'' = (pZ_2 + qZ_2')(V_1 - \sigma^2). \tag{3.63}$$

which follows from the Eqs. (3.58) and (3.60).

Thirdly we equate the coefficients of Z_2 and Z_2' on the right-hand sides of Eqs. (3.62) and (3.63), to obtain the following two equations

$$q(V_1 - \sigma^2) = 2p' + q'' + q(V_2 - \sigma^2) \tag{3.64}$$

and

$$p(V_1 - \sigma^2) = p'' + (p + 2q')(V_2 - \sigma^2) + qV_2', \tag{3.65}$$

or, alternatively,

$$q(V_1 - V_2) = 2p' + q'' \tag{3.66}$$

and

$$p(V_1 - V_2) = p'' + 2q'(V_2 - \sigma^2) + qV_2'. \tag{3.67}$$

Fourthly we proceed by eliminating $(V_1 - V_2)$ from Eqs. (3.66) and (3.67), to get

$$2pp' + pq'' - p''q - 2qq'(V_2 - \sigma^2) - q^2V_2' = 0. \tag{3.68}$$

Integrating Eq. (3.68) we get

$$p^2 + (pq' - p'q) - q^2(V_2 - \sigma^2) = \text{constant} = \mathfrak{C}^2 \quad \text{(say)}. \tag{3.69}$$

Then it turns out that Eqs. (3.66) and (3.69) are the equations which p and q must satisfy if the combination $pZ_2 + qZ_2'$ (see Eq. (3.60)) is to be a solution of Eq. (3.58).

For arbitrary V_1 and V_2 one cannot expect to solve these equations explicitly but for V_1 and V_2 of the forms specified, we can simply verify that

$$q = 2\mathfrak{A} \quad (= \text{constant}) \quad \text{and} \quad p = \mathfrak{B} + 2\mathfrak{A}^2 f, \tag{3.70}$$

do indeed satisfy Eqs. (3.66) and (3.69) with

$$\mathfrak{C}^2 = \mathfrak{B}^2 + 4\mathfrak{A}^2\sigma^2. \tag{3.71}$$

Then accordingly, Z_1 and Z_2 are related as follows

$$(\mathfrak{B} + 2i\sigma\mathfrak{A})Z_1 = (\mathfrak{B} + 2\mathfrak{A}^2 f)Z_2 + 2\mathfrak{A}Z_2', \tag{3.72}$$

where we have chosen a relative normalization of Z_1 and Z_2 such that the inverse relation in the same normalization is given by

$$(\mathfrak{B} - 2i\sigma\mathfrak{A})Z_2 = (\mathfrak{B} + 2\mathfrak{A}^2 f)Z_1 - 2\mathfrak{A}Z_1'. \tag{3.73}$$

Then finally, substituting Eqs. (3.56) and (3.57) into Eqs. (3.72) and (3.73) we get

$$[\epsilon^2(\epsilon^2+2)+12i\sigma M]Z^{(+)} = \left[\epsilon^2(\epsilon^2+2) + 72M^2\frac{\Delta}{r^3(\epsilon^2 r + 6M)}\right]Z^{(-)} + 12MZ^{(-)}_{,r_*} \tag{3.74}$$

and

$$[\epsilon^2(\epsilon^2+2)-12i\sigma M]Z^{(-)} = \left[\epsilon^2(\epsilon^2+2) + 72M^2\frac{\Delta}{r^2(\epsilon^2 r + 6M)}\right]Z^{(+)} - 12MZ^{(+)}_{,r_*}. \tag{3.75}$$

Eqs. (3.74) and (3.75) relates $Z^{(-)}$ and $Z^{(+)}$.

3.5 The Bardeen-Press equation

We use the Newman-Penrose formalism to derive the Bardeen-Press equations as follows:

We start by assuming that the perturbations have a time t and an azimuthal angle ϕ dependence given by

$$e^{i(\sigma t + m\phi)}, \tag{3.76}$$

where σ is a constant and m is an any integer. Then the directional derivatives D, $\Box$, Ξ, and Ξ^* along the basis null-vectors (see chapter 1, sec. 1.4) when acting on Eq. (3.76) become

$$l = D = \mathcal{D}_0, \quad n = \Box = -\frac{\Delta}{2r^2}\mathcal{D}_0^\dagger, \quad m = \Xi = \frac{1}{r\sqrt{2}}\mathcal{L}_0^\dagger \quad \text{and} \quad \overline{m} = \Xi^* = \frac{1}{r\sqrt{2}}\mathcal{L}_0, \tag{3.77}$$

where

$$\mathcal{D}_n = \partial_r + \frac{ir^2\sigma}{\Delta} + 2n\frac{r-M}{\Delta}, \qquad \mathcal{L}_0 = \partial_\theta + n\cot\theta + m\csc\theta, \tag{3.78}$$

$$\mathcal{D}_n^\dagger = \partial_r - \frac{ir^2\sigma}{\Delta} + 2n\frac{r-M}{\Delta}, \quad \text{and} \quad \mathcal{L}_n^\dagger = \partial_\theta + n\cot\theta - m\csc\theta. \tag{3.79}$$

$\mathcal{D}$ and $\mathcal{D}_n^\dagger$ are purely radial operators, and $\mathcal{L}_n$ and $\mathcal{L}_n^\dagger$ are purely angular operators and they are all called differential operators. They also satisfy a number of identities which are given by

$$\mathcal{D}_n^\dagger = (\mathcal{D}_n)^*; \quad \mathcal{L}_n^\dagger(\theta) = -\mathcal{L}_n(\pi-\theta) \quad \Delta\mathcal{D}_{n+1} = \mathcal{D}_n\Delta, \quad \sin\theta\mathcal{L}_{n+1} = \mathcal{L}_n\sin\theta. \tag{3.80}$$

The Weyl scalars Ψ_0, Ψ_1, Ψ_3, and Ψ_4 and the spin coefficients κ, σ, $\curlywedge$, and ς vanish in the Schwarzschild background because the Schwarzschild space-time is of Petrov type-D [25]. The only non-vanishing Weyl scalar is Ψ_2, and it is given by Eq. (2.70) and the non-vanishing spin-coefficients are give by Eqs. (2.67) and (2.68).

In section 1.4 we listed the eight components of the Bianchi identities together with the Ricci identities (see Eqs. (3.28) to (3.36) and Eqs. (3.10) to (3.27) respectively) and among them there are four Bianchi identities (Eqs. (3.28), (3.31), (3.32) and (3.36)) and two Ricci identities (Eqs. (3.11) and (3.19)) which are linear and homogeneous in the quantities Ψ_0, Ψ_1, Ψ_3, Ψ_4, κ, ϖ, $\curlywedge$, and ς. These six equations are given by

$$(\Xi^* - 4\alpha + \pi)\Psi_0 - (D - 2\varepsilon - 4\rho)\Psi_1 = 3\kappa\Psi_2, \tag{3.81}$$

$$(\Box - 4\gamma + \wp)\Psi_0 - (\Xi - 4\tau - 2\vartheta)\Psi_1 = 3\varpi\Psi_2, \tag{3.82}$$

$$(D - \rho - \rho^* - 3\varepsilon + \varepsilon^*)\varpi - (\Xi - \tau + \pi^* - \alpha^* - 3\vartheta)\kappa = \Psi_0, \tag{3.83}$$

and

$$(D+4\varepsilon-\rho)\Psi_4-(\Xi^*+4\pi+2\alpha)\Psi_3=-3\curlywedge\Psi_2, \tag{3.84}$$

$$(\Xi+4\vartheta-\tau)\Psi_4-(\Box+2\gamma+4\wp)\Psi_3=-3\nu\Psi_2, \tag{3.85}$$

$$(\Box+\wp+\wp^*+3\gamma-\gamma^*)\curlywedge-(\Xi^*+3\alpha+\vartheta^*+\pi-\tau^*)\varsigma=-\Psi_4. \tag{3.86}$$

We note that Eqs. (3.81), (3.82), (3.83), (3.84), (3.85), and (3.86) are already linearized in the sense that Ψ_0, Ψ_1, Ψ_3, Ψ_4, κ, ϖ, $\curlywedge$, and ς, are quantities of the first order of smallness with a t and ϕ dependence given by Eq. (3.76). We then replace all the other quantities including the basis vectors and the directional derivatives which occur in them by their unperturbed values given by Eqs. (3.77), (2.67), and (2.68) to get

$$\frac{1}{r\sqrt{2}}(\mathcal{L}_0+2\cot\theta)\Psi_0-\left(\mathcal{D}_0+\frac{4}{r}\right)\Psi_1=-\frac{3M}{r^3}\kappa, \tag{3.87}$$

$$-\frac{\Delta}{2r^2}\left(\mathcal{D}_0^\dagger+\frac{4(r-M)}{\Delta}-\frac{3}{r}\right)\Psi_0-\frac{1}{r\sqrt{2}}(\mathcal{L}_0^\dagger-\cot\theta)\Psi_1=-\frac{3M}{r^3}\varpi, \tag{3.88}$$

$$\left(\mathcal{D}_0+\frac{2}{r}\right)\varpi-\frac{1}{r\sqrt{2}}(\mathcal{L}_0^\dagger-\cot\theta)\kappa=\Psi_0; \tag{3.89}$$

and

$$\left(\mathcal{D}_0+\frac{1}{r}\right)\Psi_4-\frac{1}{r\sqrt{2}}(\mathcal{L}_0-\cot\theta)\Psi_3=+\frac{3M}{r^3}\curlywedge, \tag{3.90}$$

$$\frac{1}{r\sqrt{2}}(\mathcal{L}_0^\dagger+2\cot\theta)\Psi_4+\frac{\Delta}{2r^2}\left(\mathcal{D}_0^\dagger-\frac{2(r-M)}{\Delta}+\frac{6}{r}\right)\Psi_3=+\frac{3M}{r^3}\varsigma, \tag{3.91}$$

$$-\frac{\Delta}{2r^2}\left(\mathcal{D}_0^\dagger-\frac{2(r-M)}{\Delta}+\frac{4}{r}\right)\curlywedge-\frac{1}{r\sqrt{2}}(\mathcal{L}_0-\cot\theta)\varsigma=-\Psi_4. \tag{3.92}$$

We then change the above equations to symmetrical forms which are simple by writing them in terms of the following variables

$$\begin{gathered}\Phi_0=\Psi_0,\quad \Phi_1=\Psi_1 r\sqrt{2},\quad \mathfrak{k}=\frac{\kappa}{r^2\sqrt{2}},\quad \mathfrak{s}=\frac{\varpi}{r};\\ \Phi_4=\Psi_4 r^4,\quad \Phi_3=\Psi_3\frac{r^3}{\sqrt{2}},\quad \mathfrak{p}=\frac{1}{2}\curlywedge r,\quad \mathfrak{n}=\frac{1}{\sqrt{2}}\varsigma r^2,\end{gathered} \tag{3.93}$$

thereby obtaining the following set of equations

$$\mathcal{L}_2\Phi_0-\left(\mathcal{D}_0+\frac{3}{r}\right)\Phi_1=-6M\mathfrak{k}, \tag{3.94}$$

$$\Delta\left(\mathcal{D}_2^\dagger - \frac{3}{r}\right)\Phi_0 + \mathcal{L}_{-1}^\dagger\Phi_1 = +6M\mathfrak{s}, \tag{3.95}$$

$$\left(\mathcal{D}_0 + \frac{3}{r}\right)s - \mathcal{L}_{-1}^\dagger\mathfrak{k} = \frac{\Phi_0}{r}; \tag{3.96}$$

and

$$\left(\mathcal{D}_0 - \frac{3}{r}\right)\Phi_4 - \mathcal{L}_{-1}\Phi_3 = 6M\mathfrak{p}, \tag{3.97}$$

$$\mathcal{L}_2^\dagger\Phi_4 + \Delta\left(\mathcal{D}_{-1}^\dagger + \frac{3}{r}\right)\Phi_3 = 6M\mathfrak{n}, \tag{3.98}$$

$$\Delta\left(\mathcal{D}_{-1}^\dagger + \frac{3}{r}\right)l + \mathcal{L}_{-1}\mathfrak{n} = \frac{\Phi_4}{r}. \tag{3.99}$$

From Eqs. (3.94) and (3.95) we eliminate Φ_1 by multiplying Eq. (3.94) by the operator $\mathcal{L}_{-1}^\dagger$ and Eq. (3.95) by the operator $(\mathcal{D}_0 + 3/r)$ and then we add the results to get the decoupled equation

$$\left[\mathcal{L}_{-1}^\dagger\mathcal{L}_2 + \left(\mathcal{D}_0 + \frac{3}{r}\right)\Delta\left(\mathcal{D}_2^\dagger - \frac{3}{r}\right)\right]\Phi_0 = \frac{6M}{r}\Phi_0. \tag{3.100}$$

Similarly, by eliminating Φ_3 from Eqs. (3.97) to (3.99) we get the decoupled equation

$$\left[\mathcal{L}_{-1}\mathcal{L}_2^\dagger + \Delta\left(\mathcal{D}_{-1}^\dagger + \frac{3}{r}\right)\left(\mathcal{D}_0 - \frac{3}{r}\right)\right]\Phi_4 = \frac{6M}{r}\Phi_4. \tag{3.101}$$

From Eq. (3.101), the identity

$$\begin{aligned}\Delta\left(\mathcal{D}_1 + \frac{3}{r}\right)\left(\mathcal{D}_2^\dagger - \frac{3}{r}\right) - \frac{6M}{r} &= \Delta\mathcal{D}_1\mathcal{D}_2^\dagger + \frac{6}{r}[-ir^2\sigma + (r-M)] - \frac{6\Delta}{r^2} - \frac{6M}{r} \\ &= \Delta\mathcal{D}_1\mathcal{D}_2^\dagger - 6i\sigma r, \end{aligned}\tag{3.102}$$

reduces Eq. (3.100) to

$$[\mathcal{L}_{-1}^\dagger\mathcal{L}_2 + (\Delta\mathcal{D}_1\mathcal{D}_2^\dagger - 6i\sigma r)]\Phi_0 = 0. \tag{3.103}$$

Similarly, by applying the identity (3.102) to Eq. (3.101) we get

$$[\mathcal{L}_{-1}\mathcal{L}_2^\dagger + (\Delta\mathcal{D}_{-1}^\dagger\mathcal{D}_0 + 6i\sigma r]\Phi_4 = 0. \tag{3.104}$$

From here, we are now able to separate the variables r and θ in Eqs. (3.103) and (3.104) by making the following substitutions

$$\Phi_0 = \mathfrak{R}_{+2}(r)\mathfrak{S}_{+2}(\theta) \qquad \text{and} \qquad \Phi_4 = \mathfrak{R}_{-2}(r)\mathfrak{S}_{-2}(\theta), \tag{3.105}$$

where $\mathfrak{R}_{\pm 2}$ and $\mathfrak{S}_{\pm 2}$ are functions of r and θ respectively. After separating r and θ, we get the following two pairs of equations

$$\mathcal{L}^{\dagger}_{-1}\mathcal{L}_2\mathfrak{S}_{+2} = -\epsilon^2\mathfrak{S}_{+2}, \tag{3.106}$$

$$(\Delta\mathcal{D}_1\mathcal{D}^{\dagger}_2 - 6i\sigma r)\mathfrak{R}_{+2} = +\epsilon^2\mathfrak{R}_{+2}; \tag{3.107}$$

and

$$\mathcal{L}_{-1}\mathcal{L}^{\dagger}_{-2}\mathfrak{S}_{-2} = -\epsilon^2\mathfrak{S}_{-2}, \tag{3.108}$$

$$(\Delta\mathcal{D}^{\dagger}_{-1}\mathcal{D}_0 + 6i\sigma r)\mathfrak{R}_{-2} = +\epsilon^2\mathfrak{R}_{-2}, \tag{3.109}$$

where ϵ^2 is a separation constant. By expanding the equation governing $\mathfrak{S}_{+2}$, for $m = 0$, we get

$$\frac{d^2\mathfrak{S}_{+2}}{d\theta^2} + \cot\theta\frac{d\mathfrak{S}_{+2}}{d\theta} - 2(\cot^2\theta + \csc^2\theta)\mathfrak{S}_{+2} = -\epsilon^2\mathfrak{S}_{+2}. \tag{3.110}$$

Substituting $\mathfrak{S}_{+2}(\theta) = \mathbb{H}(\theta)\csc^2\theta$ in to Eq. (3.110) we get

$$\sin^3\theta\frac{d}{d\theta}\frac{1}{\sin^3\theta}\frac{d\mathbb{H}}{d\theta} + \epsilon^2\mathbb{H} = 0. \tag{3.111}$$

Then by comparing Eq. (3.111) with Eqs. (3.17) and (3.18) we note that

$$\mathbb{H}(\theta) = \mathbb{H}^{-3/2}_{l+2}(\theta) \qquad \text{and} \qquad \epsilon^2 = 2n = (l-1)(l+2). \tag{3.112}$$

Therefore,

$$\begin{aligned}\mathfrak{S}_{+2}(\theta) &= \mathbb{H}^{-3/2}_{l+2}(\theta)\csc^2\theta = \sin\theta\frac{d}{d\theta}\frac{1}{\sin\theta}\frac{dP_l(\theta)}{d\theta}\\ &= P_{l,\theta,\theta}(\theta) - P_{l,\theta}(\theta)\cot\theta \quad (m=0),\end{aligned} \tag{3.113}$$

where $P_l(\theta)$ is the Legendre function and $\mathfrak{S}_{+2}(\theta)$ becomes a spin-weighted spherical harmonic. We also note that $\Delta^2\mathfrak{R}_{+2}$ in the radial equation (3.107) satisfies the following equation

$$(\Delta\mathcal{D}_{-1}\mathcal{D}^{\dagger}_0 - 6i\sigma r)\Delta^2\mathfrak{R}_{+2} = \epsilon^2(\Delta^2\mathfrak{R}_{+2}); \tag{3.114}$$

which is the complex conjugate of Eq. (3.109) and is satisfied by $\mathfrak{R}_{-2}$. Lastly we transform Eq. (3.114) to a standard recurrent form. Firstly, we note that

$$\mathcal{D}_0 = \frac{r^2}{\Delta}\Pi_+ \qquad \text{and} \qquad \mathcal{D}^{\dagger}_0 = \frac{r^2}{\Delta}\Pi_- \tag{3.115}$$

where the operator $\Pi_\pm$ is given by

$$\Pi_\pm = \frac{d}{dr_*} \pm i\sigma \qquad \text{and} \qquad \frac{d}{dr_*} = \frac{\Delta}{r^2}\frac{d}{dr}, \tag{3.116}$$

so that from Eq. (3.114)

$$\Delta \mathcal{D}_{-1}\mathcal{D}_0^\dagger = \Delta^2 \mathcal{D}_0 \frac{1}{\Delta}\mathcal{D}_0^\dagger = r^2 \Delta \Pi_+ \left(\frac{r^2}{\Delta^2}\Pi_-\right). \tag{3.117}$$

Secondly, by replacing $\Delta^2 \mathfrak{R}_{+2}$ from Eq. (3.114) by

$$\mathbb{Y}_{+2} = r^{-3}\Delta^2 \mathfrak{R}_{+2}, \tag{3.118}$$

we are able to rewrite Eq. (3.114) in the form

$$\Pi_+ \left[\frac{r^2}{\Delta^2}\Pi_-(r^3 \mathbb{Y}_+)\right] - 6i\sigma \frac{r^2}{\Delta}\mathbb{Y}_{+2} = \epsilon^2 \frac{r}{\Delta}\mathbb{Y}_{+2}. \tag{3.119}$$

We simplify Eq. (3.119) so that we can write it in the following form

$$\Pi^2 \mathbb{Y}_{+2} + \mathbb{P}\Pi_- \mathbb{Y}_{+2} - \mathbb{Q}\mathbb{Y}_{+2} = 0, \tag{3.120}$$

where

$$\mathbb{P} = \frac{d}{dr}\log\frac{r^8}{\Delta^2} = \frac{4}{r^2}(r - 3M) \tag{3.121}$$

and

$$\mathbb{Q} = \frac{\Delta}{r^5}(\epsilon^2 r + 6M). \tag{3.122}$$

Similarly, Eq. (3.109) governing $\mathfrak{R}_{-2}$ will lead to the complex conjugate of Eq. (3.120) and by the substitution

$$\mathbb{Y}_{-2} = r^{-3}\mathfrak{R}_{-2}, \tag{3.123}$$

we get it in the form

$$\Pi^2 \mathbb{Y}_{-2} + \mathbb{P}\Pi_+ \mathbb{Y}_{-2} - \mathbb{Q}\mathbb{Y}_{-2} = 0. \tag{3.124}$$

Eqs. (3.120) and (3.124) are called the *Bardeen-Press equations* [7] and were first derived by Bardeen and Press in 1972.

3.6 Derivation of Bardeen-Press equation from Zerilli equation

In this section we prove that the Bardeen-Press equation is equivalent to the Zerilli equation by deriving it from the Zerilli equation.

We start with the component of $\Xi\Psi_0$,

$$\Xi\Psi_0 = \frac{1}{4}e^{-2\nu}[\Xi R_{0202} + \Xi R_{1212} + 2\Xi R_{0212} \\ -\Xi R_{0303} - \Xi R_{1313} - 2\Xi R_{0313)}], \tag{3.125}$$

and we rewrite it as (see Eq. (56) of [38])

$$\Xi\Psi_0 = -\frac{1}{4}e^{-2\nu}\{\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\right)(\Xi\nu - \Xi\wp_2) + \left[e^{-4\nu}\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r} + \frac{1}{r} - \nu_r\right) \\ + e^{2\nu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\right](\Xi\psi - \Xi\wp_3)\}. \tag{3.126}$$

We now separate the variables in accordance with Eqs. (3.31) and we make use of Eq. (3.38) to get

$$\Xi\Psi_0 = \frac{1}{2}\left\{i\sigma e^{-2\nu}\left[V_{,2} + \left(\frac{1}{r} - \nu_{,2}\right)V\right] - \left(\nu_{,2}V_{,2} + \sigma^2 e^{-4\nu}V + \frac{e^{-2\nu}}{r^2}N\right)\right\}e^{i\sigma t} \\ \times(P_{l,3,3} - P_{l,3}\cot\theta). \tag{3.127}$$

By suppressing the angular dependency $(P_{l,3,3}(\theta) - P_{l,3}(3)\cos\theta)$ and the time dependency factor $\frac{1}{2}e^{i\sigma t}$ and eliminating $V_{,2}$ in the second term in parenthesis on the right hand side of Eq. (3.127) by using Eq. (3.43), we are able to write Eq. (3.127) as

$$\Xi\Psi_0 = i\sigma\frac{r}{r-2M}\left[V_{,2} + \frac{r-3M}{r(r-2M)}V\right] + \sigma^2\frac{r^2}{(r-2M)^2}\left[\frac{M}{n(r-2M)}(L+X) - V\right] \\ -\frac{nr^2 - 3nMr - 3M^2}{nr^2(r-2M)^2}N - \frac{M[nr^2 - Mr(2n-1) - 3M^2]}{nr^2(r-2M)^3}L + \frac{M(r^2 - 3Mr + 3M^2)}{r^2(r-2M)^3}V. \tag{3.128}$$

We now write Eq. (3.128) in terms of $Z^{(+)}$ and $Z^{(+)}_{,r_*}$ by substituting Eqs. (3.48) and (3.51) for $Z^{(+)}$ and $Z^{(+)}_{,r_*}$ respectively into it to get

$$\Xi\Psi_0 = \frac{1}{2r(r-2M)}\left\{\frac{2n^2(n+1)r^3+6n^2Mr^2+18M^3}{r^2(nr+3M)^2}Z^{(+)}\right.$$
$$\left.+\left[2i\frac{r^2}{r-2M}+2\frac{nr^2-3nMr-3M^2}{(r-2M)(nr+3M)}\right](Z^{(+)}_{r_*}+i\sigma Z^+)\right\}. \qquad (3.129)$$

The coefficient of $Z^{(+)}$ in the first term in braces apart from a factor $r^2/(r-2m)$ is the potential V_z of Zerilli's equation and because of this fact we use the operator Π introduced above with

$$\Pi^2 = \Pi_+\Pi_- = \Pi_-\Pi_+ = d^2/dr_* + \sigma^2, \qquad (3.130)$$

so that we can write Zerilli's equation (3.52) in the form

$$\Pi^2 Z^{(+)} = V_z Z^{(+)}. \qquad (3.131)$$

Eq. (3.129) then becomes

$$\begin{aligned}\Xi\Psi_0 &= \frac{1}{2r(r-2M)}\left[\frac{r^2}{r-2M}(\Pi^2 Z^{(+)}+2i\sigma\Pi_+Z^{(+)})+2\frac{nr^2-3nMr-3M^2}{(r-2M)(nr-3M)}\Pi_+Z^{(+)}\right]\\ &= \frac{r}{2(r-2M)^2}\left[\Pi_+(\Pi_-+2i\sigma)Z^{(+)}+2\frac{nr^2-3nMr-3M^2}{r^2(nr+3M)}\Pi_+Z^{(+)}\right], \qquad (3.132)\end{aligned}$$

and simplifying it further we get

$$\Xi\Psi_0 = \frac{r}{2(r-2M)^2}\left[\Pi_+ + 2\frac{nr^2-3nMr-3M^2}{r^2(nr+3M)}\right]\Pi_+Z^{(+)}. \qquad (3.133)$$

Because of the form of Eq. (3.133), we now define a new function Y by

$$\mathbb{Y} = 2\frac{(r-2M)^2}{r}\Xi\Psi_0. \qquad (3.134)$$

Then Eq. (3.133) takes the form

$$\mathbb{Y} = \Pi_+\Pi_+Z^{(+)} + \mathbb{W}\Pi_+Z^{(+)}, \qquad (3.135)$$

where

$$\mathbb{W} = 2\frac{nr^2-3nMr-3M^2}{r^2(nr+3M)}. \qquad (3.136)$$

The equivalent form of Eq. (3.135) is

$$\mathbb{Y} = V_z Z^{(+)} + (\mathbb{W} + 2i\sigma)\Pi_+ Z^{(+)}. \tag{3.137}$$

We note that in Eq. (3.137), expressing Y in terms of $Z^{(+)}$ and $\Pi_+ Z^{(+)}$, V_z appears as the coefficient of $Z^{(+)}$.

Applying the operator Π_- to Eq. (3.135) and using Zerilli's equation (3.131), we get

$$\Pi_- \mathbb{Y} = \left(\frac{dV_z}{dr_*} + \mathbb{W}V_z\right) Z^{(+)} + \left(V_z + \frac{d\mathbb{W}}{dr_*}\right) \Pi_+ Z^{(+)}. \tag{3.138}$$

Then by defining the functions V_z and $\mathbb{W}$ as

$$\frac{dV_z}{dr_*} + \mathbb{W}V_z = -6M\frac{(r-2M)^2}{r^6} \tag{3.139}$$

and

$$\frac{d\mathbb{W}}{dr_*} + V_z = 2\left(1 - \frac{2M}{r}\right)\frac{nr + 3M}{r^3}, \tag{3.140}$$

Eq. (3.138) takes the form

$$\Pi_- \mathbb{Y} = -6M\frac{(r-2M)^2}{r^6} Z^{(+)} + 2\left(1 - \frac{2M}{r}\right)\frac{nr + 3M}{r^3}\Pi_+ Z^{(+)}. \tag{3.141}$$

Using the relation[3]

$$\frac{r^2}{r-2M}(nr + 3M)V_z + 3M\mathbb{W} = 2n(n+1), \tag{3.142}$$

Eqs. (3.137) and (3.141) can be solved for $Z^{(+)}$ and $\Pi_+ Z^{(+)}$ to get

$$\left[\frac{2}{3}n(n+1) + 2Mi\sigma\right] Z = \frac{r^2(nr+3M)}{3(r-2M)}\mathbb{Y} - \frac{r^6}{6(r-2M)^2}(\mathbb{W} + 2i\sigma)\Pi_- Y \tag{3.143}$$

and

$$\left[\frac{2}{3}n(n+1) + 2Mi\sigma\right] \Pi_+ Z^{(+)} = M\mathbb{Y} + \frac{r^6}{6(r-2M)^2} V_z \Pi_- \mathbb{Y}. \tag{3.144}$$

By eliminating Z from Eqs. (3.143) and (3.144) we get an equation for Y given by

$$\begin{aligned} \Pi_- \left[\frac{r^6}{6(r-2M)^2} V_z \Pi_- \mathbb{Y} + M\mathbb{Y}\right] &= \left[\frac{2}{3}n(n+1) + 2Mi\sigma\right] \Pi^2 Z \\ &= \left[\frac{2}{3}n(n+1) + 2Mi\sigma\right] V_z Z, \end{aligned} \tag{3.145}$$

[3]Functions V_z and $\mathbb{W}$ satisfy three simple relations such as (3.139), (3.140) and (3.144), and as such they dispels the 'mystery' about them.

In accordance with Eq. (3.143), we conclude that

$$\Pi_- \left[\frac{r^6}{6(r-2M)^2} V_z \Pi_- \mathbb{Y} + M\mathbb{Y}\right] = V_z \left[\frac{r^2(nr-3M)}{3(r-2M)}\mathbb{Y} - \frac{r^6}{6(r-2M)^2}(\mathbb{W} + 2Mi\sigma)\Pi_- \mathbb{Y}\right]. \tag{3.146}$$

Then finally further simplifying the above equation further we get

$$\Pi^2 \mathbb{Y} + 4\frac{r-3M}{r^2}\Pi_- \mathbb{Y} - 2\left(1 - \frac{2M}{r}\right)\frac{nr+3M}{r^3}\mathbb{Y} = 0. \tag{3.147}$$

By making the final substitution

$$\mathbb{Y} = \frac{r-2M}{r^3}\phi, \tag{3.148}$$

we recover the Bardeen-Press equation that we wanted to derive from the Zerilli equation (3.131).

Conversely: we are able to derive the Zerilli equation from Eqs (3.137) and (3.141) together with the Bardeen-Press equation (3.147).

3.7 Derivation of Bardeen-Press equation from Regge-Wheeler equation

In this section we prove that Bardeen-Press equation is equivalent to Regge-Wheeler equation by deriving it from Regge-Wheeler equation.

We start by separating the variables r and θ in Eq. (3.15) by making the substitution

$$Q = rX(r)P_{l+2}(\cos\theta/-3), \tag{3.149}$$

where $P_{l+2}(x/-3)$ is the Gegenbauer polynomial of order $(l+2)$ and index -3. The above substitution reduces Eq. (3.15) to Regge-Wheeler equation (3.24)

$$\Pi^2 Z^{(-)} = V_0 Z^{(-)}, \tag{3.150}$$

where

$$V_0 = 2\left(1 - \frac{2M}{r}\right)\frac{(n+1)r - 3M}{r^3}. \tag{3.151}$$

By defining

$$\mathbb{W}_0 = 2\frac{r-2M}{r^6}, \tag{3.152}$$

we are able to verify that the functions V_0 and $\mathbb{W}_0$ satisfy the identities (see [24])

$$\frac{dV_0}{dr_*} + \mathbb{W}_0 V_0 = 6M\frac{(r-2M)^2}{r^6}, \tag{3.153}$$

$$\frac{d\mathbb{W}_0}{dr_*} + V_0 = 2\left(1-\frac{2M}{r}\right)\frac{nr+3M}{r^3} \tag{3.154}$$

and

$$\frac{r^2}{r-2M}(nr+3M)V_0 - 3M\mathbb{W}_0 = 2n(n+1). \tag{3.155}$$

The above identities are similar to those satisfied in the case of Zerilli's equation (see Eqs. (3.139), (3.140) and (3.142)).

We now let

$$\mathbb{Y} = \Pi_+\Pi_+ Z^{(-)} + \mathbb{W}_0\Pi_+ Z^{(-)}, \tag{3.156}$$

or equivalently

$$\mathbb{Y} = V_0 Z^{(-)} + (\mathbb{W}_0 + 2i\sigma)\Pi_+ Z^{(-)}, \tag{3.157}$$

and by virtue of Eqs. (3.153) and (3.154) we have

$$\Pi_-\mathbb{Y} = 6M\frac{(r-2M)^2}{r^6}Z^{(-)} + 2\left(1-\frac{2M}{r}\right)\frac{nr+3M}{r^3}\Pi_+ Z^{(-)}. \tag{3.158}$$

Then solving Eqs. (3.156), (3.157) and (3.158) for $Z^{(-)}$ and $\Pi_+ Z^{(-)}$, we get,

$$\left[\frac{2}{3}n(n+1) - 2Mi\sigma\right]Z^{(-)} = \frac{r^2(nr+3M)}{3(r-2M)}\mathbb{Y} - \frac{r^6}{6(r-2M)^2}(\mathbb{W}_0 + 2i\sigma)\Pi_-\mathbb{Y} \tag{3.159}$$

and

$$\left[\frac{2}{3}n(n+1) - 2Mi\sigma\right]\Pi_+ Z^{(-)} = -M\mathbb{Y} + \frac{r^6}{6(r-2M)^2}V_0\Pi_-\mathbb{Y}. \tag{3.160}$$

Then finally, Eqs. (3.157), (3.158), (3.159) and (3.160) are necessary and sufficient for the Regge-Wheeler equation (3.150) to imply the Bardeen-Press equation (3.120) or (3.124) and *conversely.*

3.8 Schwarzschild quasi-normal modes

Complex resonant frequencies which are characteristic of the Schwarzschild geometry were first discovered by Vishveshwara [90] when he was doing the calculations of the scattering of gravitational waves by black holes. Recent speculation as to the role that black holes might play in a variety of astrophysical processes has created considerable interest in the methods of computing these resonant (or quasi-normal) frequencies. We define quasi-normal modes as solutions of the linear perturbation equations (i.e. Zerilli and Regge-Wheeler equations), belonging to complex characteristic-frequencies σ with $Re(\sigma) \geq 0$ and satisfying the boundary conditions appropriate for purely outgoing waves at $+\infty$ and purely ingoing waves at the $-\infty$ (horizon). The problem then is to seek solutions of the equations governing $Z^{(\pm)}$ which will satisfy the boundary conditions

$$
\begin{aligned}
Z^{(\pm)} \quad &\rightarrow \quad \mathcal{A}^{(\pm)}(\sigma)e^{(-i\sigma\hat{r}_*)} \quad (\hat{r}_* \rightarrow +\infty) &(3.161)\\
&\rightarrow \quad e^{+i\sigma\hat{r}_*} \quad (\hat{r}_* \rightarrow -\infty) &(3.162)
\end{aligned}
$$

where $Z^{(-)}$ represents the outgoing wave and $Z^{(+)}$ represents the ingoing wave, $\mathcal{A}^{(\pm)}(\sigma)e^{(-i\sigma\hat{r}_*)}$ and $e^{(+i\sigma\hat{r}_*)}$ are reflected and transmitted waves respectively, and $\mathcal{A}^{(\pm)}$ are amplitudes.

The quasi-normal modes that belongs to the odd-parity perturbations are called the odd-parity modes and the ones that belong to the even-parity perturbations are called the even-parity modes. In this section we outline the analytic method employed by Leaver [55] and the numeric integration method employed by Chandrasekhar and Detwieler [22] in the form of the linearized boundary-value problem on a stationary Schwarzschild black hole background. Then lastly we end this section by outlining new recent developments by Fiziev [36] on quasi-normal modes of Schwarzschild black hole.

3.8.1 Numerical integration method: by Chandrasekhar

Chandrasekhar and Detweiler solved the Schwarzschild quasi-normal modes problem by employing a numerical integration scheme to solve the separated partial differential equation with sufficient accuracy to allow the determination of the under-damped ($i.e.|Re(\sigma)| > |Im(\sigma)|$) quasi-normal frequencies [22]. Difficulties with integration methods are discussed by [29].

Theory

We consider a simple one dimensional wave equation given by

$$\frac{d^2\Upsilon}{dx^2} + [\sigma^2 - V(x)]\Upsilon = 0 \quad (-\infty < x < +\infty), \tag{3.163}$$

where Υ is a wave function and $V(x)$ is positive everywhere and is of 'short range' in the sense that

$$\int_{-}^{+} V(x)dx \quad \text{is finite.} \tag{3.164}$$

We suppose that we have a plane wave of unit amplitude $e^{+i\sigma x}$ incident on the barrier from the right[4]. Then part of it will be reflected back and part of it will be transmitted through the barrier. The reflected wave $\mathcal{A}e^{-i\sigma x}$ will be at $+\infty$ and the transmitted wave $\mathcal{B}e^{i\sigma x}$ will be at $-\infty$. So long as σ is real, the reflection and transmission coefficients of $\mathcal{A}e^{-i\sigma x}$ and $\mathcal{B}e^{i\sigma x}$ are respectively given by

$$\mathbb{R} = |\mathcal{A}^2| \quad \text{and} \quad \mathbb{T} = |\mathcal{B}|^2, \tag{3.165}$$

where

$$\mathbb{R} + \mathbb{T} = 1 \quad \text{(always true)}. \tag{3.166}$$

Then we transform Eq. (3.163) to the form of the Riccati equation by making use of the substitution,

$$\Upsilon = \exp\left(i\int^x \varphi dx\right) \quad (\varphi \text{ is the phase function}), \tag{3.167}$$

[4] The convention that $e^{+i\sigma t}$ represent an ingoing wave is the opposite of the one which is normally adopted in the quantum theory; it is a consequence of the assumption, normal in this theory, that the time-dependence of the normal modes is $e^{i\sigma t}$ [22, 24].

and after the substitution we get

$$id\varphi/dx + \sigma^2 - \varphi^2 - V(x) = 0. \tag{3.168}$$

We make this sort of transformation because in the case of general potential barriers (i.e. Zerilli's potential), it is not easy to find explicit solutions. According to Eq. (3.168) a quasi-normal mode will correspond to a solution which satisfies the boundary conditions

$$\varphi \to -\sigma \quad \text{as} \quad x \to +\infty \quad \text{and} \quad \varphi \to +\sigma \quad \text{as} \quad x \to -\infty,, \tag{3.169}$$

with the real part of σ assumed to be positive. Generally speaking, the solutions with the properties (3.169) exist when σ assumes one of a discrete set of complex values and the set need not be an enumerable infinity, although sometimes it can but not always [23].

Now integrating Eq. (3.168) over the entire range of x and making use of the boundary condition Eq. (3.169), we get the following identity

$$-2i\sigma + \int_{-\infty}^{+\infty} (\sigma^2 - \varphi^2)dx = \int_{-\infty}^{+\infty} V(x)dx. \tag{3.170}$$

From the above identity we note that the two integrals are finite because of the boundary conditions (3.169) and the assumed short range character of $V(x)$ in Eq. (3.163).

We then separate the real and the imaginary parts of Eq. (3.168) by writing

$$\sigma = \sigma_1 + i\sigma_2 \quad \textit{and} \quad \varphi = \varphi_1 + i\varphi_2 \quad (\sigma_1 \geq 0). \tag{3.171}$$

so that we can obtain the following pair of equations

$$d\varphi_1/dx = -2\sigma_1\sigma_2 + 2\varphi_1\varphi_2 \tag{3.172}$$

and

$$d\varphi_2/dx = \sigma_1^2 - \sigma_2^2 - \varphi_1^2 + \varphi_2^2 - V, \tag{3.173}$$

with the boundary equations

$$\varphi_1 \to \sigma_1 \quad \text{as} \quad x \to +\infty \quad \text{and} \quad \varphi_1 \to +\sigma_1 \quad \text{as} \quad x \to -\infty, \tag{3.174}$$

and

$$\varphi_2 \to -\sigma_2 \quad \text{as} \quad x \to +\infty \quad \text{and} \quad \varphi_2 \to +\sigma_2 \quad \text{as} \quad x \to -\infty. \tag{3.175}$$

If σ is purely imaginary then $\sigma_1 = 0$. In our case φ must vanish at $\pm\infty$ and it follows from Eq. (3.172) that $\varphi_1 \equiv 0$. Finally, we are left with finding the existence of non-trivial solutions of the equation,

$$d\varphi_2/dx = -\sigma_2^2 + \sigma_2^2 - V, \tag{3.176}$$

which satisfy the boundary conditions Eq. (3.175).

Even-parity modes

In determining even-parity modes described by the Zerilli equation (Eq. (3.52)), we note that its potential (3.53) is clearly of short range, and that it has the integral property given by [21]

$$2M \int_{-\infty}^{+\infty} V_z d\hat{r}_* = 2n + \frac{1}{2} = (l-1)(l+2) + \frac{1}{2}. \tag{3.177}$$

As $\hat{r}_* \to \pm\infty$, Eq. (3.52) has two independent solutions with the asymptotic behaviour

$$Z_{\pm} \to e^{\pm i\sigma\hat{r}_*}, \tag{3.178}$$

The required series for Eq. (3.52) are of the form

$$Z^{(+)} = e^{-i\sigma\hat{r}_*} \sum_{j=0}^{\infty} \mathbb{L}_j r^{-j} \quad (\hat{r}_* \to +\infty) \tag{3.179}$$

and

$$Z^{(+)} = e^{+i\sigma\hat{r}_*} \sum_{j=0}^{\infty} \mathbb{M}_j (r-2M)^j \quad (\hat{r}_* \to \infty), \tag{3.180}$$

where $\mathbb{L}_j$ and $\mathbb{M}_j$ are coefficients, and they are determined with the help of the following recurrence relations [22, 24]

$$\begin{aligned}2i\sigma n^2(j+1)\mathbb{L}_{j+1} &+ [n^2j(j+1) - 2n^2(n+1) + 12i\sigma Mjn]\mathbb{L}_j \\ &+ M[6nj(j-1) - 2n^2(j^2-1) - 6n^2 + 18i\sigma M(j-1)]\mathbb{L}_{j-1} \\ &+ M^2[9(j-1)(j-2) - 12nj(j-2) - 18n]\mathbb{L}_{j-2} \\ &- 18M^3[(j-1)(j-3)+1]\mathbb{L}_{j-3} = 0, \end{aligned} \tag{3.181}$$

and

$$\begin{aligned}2i\sigma n^2(j-1)\mathbb{M}_{j-1} + j[n^2(j-1+8i\sigma M) + 12i\sigma Mn(n+1) - \mathbb{A}/j)]\mathbb{M}_j \\ +M(j+1)[n^2(2j+2+8i\sigma M) + 6n(n+1)(j+8i\sigma M) \\ +6i\sigma M(2n+1)(2n+3) - (\mathbb{B}/M(j+1))]\mathbb{M}_{j+1} \\ +M^2(j+2)[6n(n+1)(2j+4+8i\sigma M) + 3(2n+1)(2n+3)(j+1+8i\sigma M) \\ +4i\sigma M(2n+3)^2 - (\mathbb{C}/M^2(j+2))]\mathbb{M}_{j+2} \\ +M^3(j+3)[3(2n+1)(2n+3)(2j+6+8i\sigma M) + 2(2n+3)^2(j+2+8i\sigma M) \\ -(\mathbb{D}/M^3(j+3))]\mathbb{M}_{j+3} \\ +4M^4(j+4)(2n+3)^2(j+4+4i\sigma M)\mathbb{M}_{j+4} = 0,\end{aligned} \tag{3.182}$$

where

$$\begin{aligned}\mathbb{A} &= 2n^2(n+1), \quad \mathbb{B} = 6n^2(2n+3)M, \\ \mathbb{C} &= 6n(4n^2+8n+3)M^3 \quad \text{and} \quad \mathbb{D} = (16n^3+40n^2+36n+18)M^3.\end{aligned} \tag{3.183}$$

From the series expansions (3.179) and (3.180), we are able to determine $Z^{(+)}$ at $\pm\infty$, and φ follows from

$$\varphi = \frac{1}{Z^{(+)}}\frac{dZ^{(+)}}{d\hat{r}_*}. \tag{3.184}$$

By choosing a complex value of σ in the positive half of the plane ($Re(\sigma) > 0$) and from Eqs. (3.181) and (3.183) we are able to determine the expansion coefficients

α_j and β_j. We use the series expansions (3.179) and (3.180) to evaluate φ accurately enough[5] for values of $\hat{r}_*$ (both positive and negative).

Lastly, we integrate backward numerically from $+\infty$ and forward from $-\infty$ to a common intermediate value of $\hat{r}_*$ (generally $3M$ where V_z is approximately at its maximum). We find at this common value the difference

$$M(\sigma) = \varphi_-(\hat{r}_*) - \varphi_+(\hat{r}_*). \tag{3.185}$$

Because of the Riccati equation (Eq. (3.168)) is of the first order, the condition that the chosen value of σ belongs to a quasi-normal mode is that $M(\sigma)$ vanishes.

With the above procedure we are able to determine quasi-normal modes as long as

$$|Im(\sigma)| \leq |Re(\sigma)|. \tag{3.186}$$

If the above condition is violated, then the numerical integration will suffer from instabilities. In the table below we list the complex frequencies of the quasi normal modes of Zerilli's potential.

l	$2M\sigma$	l	$2M\sigma$
2	$0.74734+0.17792i$	4	$1.61835+0.18832i$
	$0.69687+0.54938i$		$1.59313+0.56877i$
3	$1.19889+0.18541i$		$1.12019+0.84658i$
	$1.16402+0.56231i$	5	$2.02458+0.18974i$
	$0.85257+0.74546i$	6	$2.42402+0.19053i$

Table 3.2: The complex characteristic-frequencies belonging to the quasi-normal modes of Zerilli's potentials: σ is expressed in the units $(2M)^{-1}$.

The entries in the different lines for l = 2, 3, 4, 5 and 6 corresponds to the characteristic values belonging to different modes.

[5]It is necessary to retain as many terms in the expansions as they are necessary to determine φ until it is substantially different from its limiting values at $\pm\infty$.

Odd-parity modes

It is well known in the literature that odd-parity modes are described by the Regge-Wheeler equation (Eq. (3.24)) [70]. In this section we show that the Regge-Wheeler equation actually yields the same complex frequencies and the same reflection and transmission coefficients as Zerilli's equation.

We start by rewriting Regge-Wheeler equation (Eq. (3.24)) as

$$d^2Z^{(-)}/d\hat{r}_*^2 + (\sigma^2 - V_0)Z^{(-)} = 0, \tag{3.187}$$

where

$$V_0 = 2\left(1 - \frac{2M}{r}\right)\frac{(n+1)r - 3M}{r^3}, \tag{3.188}$$

and we note that the integral of V_0 over the range of $\hat{r}_*$ has the same value as Eq. (3.177) for the Zerilli equation.

Then substituting Eqs. (3.137) and (3.138) in Eq. (3.159) and after simplification we get the final result

$$\left[\frac{2}{3}n(n+1) - 2Mi\sigma\right]Z^{(-)} = \left[\frac{2}{3}n(n+1) + \frac{6M^2(r-2M)}{r^2(nr+3M)}\right]Z^{(+)} - 2M\frac{dZ^{(+)}}{d\hat{r}_*}. \tag{3.189}$$

Therefore from Eq. (3.189) we see that a solution of Zerilli's equation with the asymptotic behavior

$$\begin{aligned} Z^{(+)} &\rightarrow e^{+i\sigma\hat{r}_*} + \mathcal{A}e^{-i\sigma\hat{r}_*} \quad (\hat{r}_* \rightarrow +\infty) \\ &\rightarrow \mathcal{B}e^{+i\sigma\hat{r}_*} \quad (\hat{r}_* \rightarrow -\infty) \end{aligned} \tag{3.190}$$

will yield a solution of Eq. (3.187) with the behavior

$$\begin{aligned} Z^{(-)} &\rightarrow e^{+i\sigma\hat{r}_*} + \frac{\frac{2}{3}n(n+1) + 2Mi\sigma}{\frac{2}{3}n(n+1) - 2Mi\sigma}\mathcal{A}e^{-i\sigma\hat{r}_*} \quad (\hat{r}_* \rightarrow +\infty) \\ &\rightarrow \mathcal{B}e^{+i\sigma\hat{r}_*} \quad (\hat{r}_* \rightarrow -\infty). \end{aligned} \tag{3.191}$$

Hence we have shown the equality between the reflection and transmission coefficients that are determined by both Zerilli and Regge-Wheeler equations. Vishveshwara [90] also suspected this equality from the numerical agreement he had found from a direct evaluation of these coefficients. Also from above, it is clear that complex frequencies belonging to quasi-normal modes of even and odd-parities perturbations must be the same and quasi-normal modes must be related by Eq. (3.189).

3.8.2 Analytic method: by Leaver

The method presented here is similar to the original one above, but uses analytic solutions to the differential equation and it allows a complete characterization of the quasi-normal frequencies for a Schwarzschild black hole.

Given the Schwarzschild coordinates (t, r, θ, ϕ), we let $\mathcal{H}(t, r, \theta, \phi)$ denote a component of a linear perturbation to a massless spin s field. In fact, Wheeler [94], Regge and Wheeler [70], Zerilli [99], Bardeen and Press [7], Chandrasekhar [21], and Chandrasekhar and Detweiler [22], all studied the wave equations obeyed by $\mathcal{H}$. We suppose that $\mathcal{H}(t, r, \theta, \phi)$ is Fourier analyzed and expanded in spherical harmonics as

$$\mathcal{H}(t, r, \theta, \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\sigma t} \left(\sum_{l} \frac{1}{r} \mathcal{H}_l(r, \sigma) Y_{lm}(\theta, \varphi) \right) d\sigma, \tag{3.192}$$

($Y_{lm}(\theta, \phi)$ are spherical harmonics.)
so that we can write the ordinary differential equation satisfied by $\mathcal{H}_l(r, \sigma)$ in the form where t and r are scaled such that $c = G = 2M = 1$ as

$$r(r-1)\mathcal{H}_{l,1,1} + \mathcal{H}_{l,1} + \left[\frac{\sigma^3 r^3}{r-1} - l(l+1) + \frac{\epsilon}{r} \right] \mathcal{H}_l = 0. \tag{3.193}$$

The index ϵ in the third term in the square brackets in Eq. (3.193) is one less than the square of the field's spin and it takes the values -1, 0, or +3 depending on whether $\mathcal{H}$ is representing a component of a scalar, electromagnetic, or gravitational field respectively.

We note that Eq. (3.193) is a second order differential equation and has two regular singular points and one irregular singular point and that it belongs to a class of differential equations known as *generalized spheroidal wave equations* [95]. The locations of the regular singular points are at $r = 0$ and at the event horizon given by $r = 1$ while the location of the irregular singular is at $r = \infty$. The singular point at $r = 0$ has indices of $1 \pm (\epsilon + 1)^{\frac{1}{2}}$, and the singular point at $r = 1$ has indices $\pm i\sigma$.

Eq. (3.193) has the asymptotic solutions given by $\mathcal{H}_l \rightarrow \exp[\pm i\sigma(r+\log r)]$ as $r \rightarrow \infty$ and the boundary conditions for the exterior eigenvalue problem (the quasi-normal mode problem) are given by $\mathcal{H}_l \rightarrow (r-1)^{i\sigma}$ as $r \rightarrow 1$, and $\mathcal{H}_l \rightarrow \exp[-i\sigma(r+\log r)]$ as $r \rightarrow \infty$. The effect of these boundary conditions is that they ensure that the field radiates only inward at the horizon and only outward at spatial infinity.

Because of the notation, we introduce a new frequency variable ϱ defined by $\varrho = i\sigma$ so that we can express Eq. (3.193) (the boundary value problem) as the differential equation

$$r(r-1)\mathcal{H}_{l,1,1} + \mathcal{H}_{l,1} - \left[\frac{\varrho^2 r^3}{r-1} + l(l+1) - \frac{\epsilon}{r}\right]\mathcal{H}_l = 0 \qquad (3.194)$$

with boundary conditions given by

$$\mathcal{H}_l \rightarrow (r-1)^{\varrho} \quad \text{as} \quad r \rightarrow 1 \quad \text{and} \quad \mathcal{H}_l \rightarrow r^{-\varrho}e^{-\varrho r} \quad \text{as} \quad r \rightarrow \infty. \qquad (3.195)$$

Then we write a solution to Eq. (3.194) at the event horizon $(r-1)$ in the form [4]

$$\mathcal{H}_l = (r-1)^{\varrho} r^{-2\varrho} e^{-\varrho(r-1)} \sum_{n=0}^{\infty} a_n \left(\frac{r-1}{r}\right)^n. \qquad (3.196)$$

The sequence of expansion coefficient $\{a_n : n = 1, 2...\}$ is determined by a three-term recurrence relation starting with $a_0 = 1$:

$$\mathfrak{u}_0 a_1 + \mathfrak{w}_0 a_0 = 0, \qquad (3.197)$$

$$\mathfrak{u} a_{n+1} + \mathfrak{w}_n a_n + \mathfrak{z}_n a_{n-1} = 0, \qquad n = 1, 2.... \qquad (3.198)$$

where the recurrence coefficients $\mathfrak{u}_n$, $\mathfrak{w}_n$, and $\mathfrak{z}_n$ are simple functions of n and they are given by

$$\begin{aligned}
\mathfrak{u} &= n^2 + (2\varrho + 2)n + 2\varrho + 1, \\
\mathfrak{w}_n &= -(2n^2 + (8\varrho + 2)n + 8\varrho^2 + 4\varrho + l(l+1) - \epsilon), \\
\mathfrak{z}_n &= n^2 + 4\varrho n + 4\varrho^2 - \epsilon - 1. \qquad (3.199)
\end{aligned}$$

At spatial infinity the boundary condition will be satisfied for those values of $\sigma = \sigma_n$ (quasi-normal frequencies) for which the series in Eq. (3.196) is absolutely convergent (by absolute convergent we meant Σa_a exists and is finite).

We may use the theory of three-term recurrence relations [40] to determine the conditions under which this sum of coefficients converges. The analysis by [4] for the behavior of the expansion coefficients a_n for large n indicates that

$$\frac{a_{n+1}}{a_n} \to 1 \pm \frac{(2\varrho)^{\frac{1}{2}}}{n^{\frac{1}{2}}} + \frac{2\varrho - \frac{3}{4}}{n} + \cdots \quad \text{as} \quad n \to \infty. \tag{3.200}$$

We note that the series in Eq. (3.196) will only converge uniformly if we consider the minus sign in Eq. (3.200), which will happen only for eigenvalues ϱ corresponding to quasi-normal frequencies. From Eq. (3.200) the a_n now form a 'solution sequence to the recurrence relation Eq. (3.198) that is minimal as $n \to \infty$' [40]. The ratio of successive a_n is given by the infinite continued fraction

$$\frac{a_{n+1}}{a_n} = \frac{-\mathfrak{z}_{n+1}}{\mathfrak{w}_{n+1} - \frac{\mathfrak{u}_{n+1}\mathfrak{z}_{n+2}}{\mathfrak{w}_{n+2} - \frac{\mathfrak{u}_{n+2}\mathfrak{z}_{n+3}}{\mathfrak{w}_{n+3} - \cdots}}}. \tag{3.201}$$

The usual notation for such a continued fraction is

$$\frac{a_{n+1}}{a_n} = \frac{-\mathfrak{z}_{n+1}}{\mathfrak{w}_{n+1}-} \frac{a_{n+1}\mathfrak{z}_{n+1}}{\mathfrak{w}_{n+2}-} \frac{a_{n+2}\mathfrak{z}_{n+3}}{\mathfrak{w}_{n+3}-} \cdots \tag{3.202}$$

In order to understand Eq. (3.202), we may think of it as an '$n = \infty$ boundary condition' on the sequence a_n [55]. By evaluating Eq. (3.202) at $n = 0$, and using equation Eq. (3.197) as an '$n = 0$ boundary condition' on the ratio a_1/a_0, we get a characteristic equation for the quasi-normal frequencies. To be specific, we have two equations that must be satisfied which are given by

$$\frac{a_1}{a_0} = \frac{\mathfrak{w}_0}{\mathfrak{u}_0} \tag{3.203}$$

and

$$\frac{a_1}{a_0} = \frac{-\mathfrak{z}_1}{\mathfrak{w}_1-} \frac{\mathfrak{u}_1\mathfrak{z}_2}{\mathfrak{w}_2-} \frac{\mathfrak{u}_2\mathfrak{z}_3}{\mathfrak{w}_3-} \cdots \tag{3.204}$$

By equating the right-hand side of Eqs. (3.203) and (3.204) we get the desired characteristic equation implicitly for quasi-normal frequencies given by

$$0 = \mathfrak{w}_0 - \frac{\mathfrak{u}_0\mathfrak{z}_1}{\mathfrak{w}_1-} \frac{\mathfrak{u}_1\mathfrak{z}_2}{\mathfrak{w}_2-} \frac{\mathfrak{u}_2\mathfrak{z}_3}{\mathfrak{w}_3-} \cdots. \tag{3.205}$$

One must note that the $\mathfrak{u}_n$, $\mathfrak{w}_n$, and $\mathfrak{z}_n$ are explicit functions of the frequency $\varrho = -i\sigma$, and are given by Eq. (3.198).

On inverting Eq. (3.205) an arbitrary number of times, we get an equality between two continued fractions, one of infinite length as in Eq. (3.205) and the other of finite length given by

$$\left[\mathfrak{w}_n - \frac{\mathfrak{u}_{n-1}\mathfrak{z}_n}{\mathfrak{w}_{n-1}-}\frac{\mathfrak{u}_{n-2}\mathfrak{z}_{n-1}}{\mathfrak{w}_{n-2}-}\cdots - \frac{\mathfrak{u}_0\mathfrak{z}_1}{\mathfrak{w}_0}\right] = \left[\frac{\mathfrak{u}_n\mathfrak{z}_{n+1}}{\mathfrak{w}_{n+1}-}\frac{\mathfrak{u}_{n+1}\mathfrak{z}_{n+2}}{\mathfrak{w}_{n+2}-}\frac{\mathfrak{u}_{n+2}\mathfrak{z}_{n+3}}{\mathfrak{w}_{n+3}-}\cdots\right] \tag{3.206}$$

where $(n = 1, 2...)$.

For every $n > 0$, Eqs. (3.205) and (3.206) are completely equivalent because a solution of Eq. (3.205) is also a solution of (3.206), and vice versa. We can take any of these two equations as the defining equation for the Schwarzschild quasi-normal frequencies σ_n, and the problem of determining these frequencies is now reduced to the numerical problem of finding the roots of this equations. When we look at Eq. (3.205), we immediately suspect that it should have infinite number of roots. Although Lehner [55] had no formal proof of this infinite number of roots, the proof was finally given rigorously by Bachelot *et al.* [5] in 1993. In the table below, he calculated the first sixty roots for $l = 2$.

n	$\sigma_n\ (l=2)$	n	$\sigma_n\ (l=2)$
1	(0.747343,+0.177925i)	11	0.153107,+5.121653i
2	(0.693422,+0.547830i)	12	0.165196,+5.630885i
3	(0.602107,+0.956554i)	20	0.175608,+9.660879i
4	(0.503010,+1.410296i)	30	0.165814,+14.677118i
5	(0.415029,+1.893690i)	40	0.156368,+19.684873i
6	(0.338599,+2.391216i)	41	0.154912,+20.188298i
7	(0.266505,+2.895822i)	42	0.156392,+20.685530i
8	(0.185617,+3.407676i)	50	0.151216,+24.693716i
9	(0.000000,+3.998000i)	60	0.148484,+29.696417i
10	(0.126527,+4.605289i)		

Table 3.3: Schwarzschild quasi-normal frequencies for $l = 2$

From the table above, for $n = 1$ (the fundamental mode) the complex frequency is $\sigma \approx (0.74734, +0.17792i)$, which for a $10M_{\odot}$ it corresponds to a frequency of 2.4 kHz and a damping time of 1.1 ms [52].

3.8.3 Recent Method: by Fiziew

Fiziew [36] recently determined for the first time, the exact solutions of Regge-Wheeler equation using Heun's equations [28]. He used these exact solutions to develop a new simple technique to numerically calculate quasi-normal modes of a Schwarzschild black hole and spherically symmetric massive objects using a code written in a Maple 10 package. To verify his method, he compared his results with the ones obtained numerically by Chandrasekhar and Detwieler [22], Leaver [55] and Anderson [2] earlier on.

Fiziew's new method overcome the difficulties posed by Chandrasekhar-Detwieler method, and it produces results which are in agreement with the ones produced by Leaver and Anderson methods. Since the Leaver and Anderson methods provide the most known accurate values for quasi-normal modes of a Schwarzschild black hole, it follows that the Fiziew method is correct.

3.9 Conclusion

In this chapter, we used Maple to recalculate the Ricci tensors for non-stationary axisymmetric space-times and we recovered some common factors and nonlinear terms that did not appear in the original results by Chandrasekhar. We also noted the fact that these errors do not affect his work on linear perturbations of a Schwarzschild black hole. We then used these Recci tensors to derive both Zerilli and Regge-Wheeler equations and shown how they are related via their potentials: $V^{(+)}$ and $V^{(-)}$ respectively, and also via their one-dimensional Schrodinger wave functions: $Z^{(+)}$ and $Z^{(-)}$ respectively. We also derived the Bardeen-Press equation using the Newman-Penrose formalism and showed how it is related to both Zerilli and Regge-Wheeler equations. Lastly we introduced the concept of quasi-normal modes of a Schwarzschild black hole. The next chapter attempts to connect the analytic work of this chapter with the work of chapter 5 where we shall be transforming the work of Chandrasekhar to the regime of numerical relativity by transforming his results on linear perturbations of a Schwarzschild black hole into Bondi-Sachs framework.

Chapter 4

Bondi-Sachs metric

4.1 Introduction

In chapter 1 we introduced an overview of the Bondi-Sachs metric in general terms, in this chapter we introduce Bondi-Sachs metric concepts that are relevant for the purpose of this dissertation, and in the process we introduce complex quantities U and J that are needed for the next chapter. We also introduce linearized Einstein vacuum equations for Bondi-Sachs metric and then we reduce these equations to ordinary differential equations and solve them to obtain a linearized solution which is needed for the next chapter. Then lastly we linearize Bondi-Sachs metric when the angular momentum (l) is 2 about Schwarzschild background. The results obtained are also needed for the next chapter.

4.2 The Bondi-Sachs metric

We use coordinates based upon a family of outgoing null hypersurfaces u = const. where u is the retarded time parameter. We let x^A ($A = 2, 3$) be the null rays, and r be a surface area coordinate. In the resulting $x^i = (u, r, x^A)$ coordinates, the metric takes the BS form [16, 74]

$$ds^2 = - \left[e^{2\beta}\left(1 + \frac{W}{r}\right) - r^2 h_{AB} U^A U^B\right] du^2 - 2e^{2\beta} du dr$$

$$- \quad 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B, \tag{4.1}$$

where $h^{AB} h_{BC} = \delta^A_B$ and $det(h_{AB}) = det(q_{AB})$, with q_{AB} being a unit sphere metric.

We work in spherical polar coordinates $x^A = (\theta, \phi)$ and the unit sphere metric is given by

$$\begin{aligned} q_{AB} dx^A dx^B &= g_{22} dx^2 dx^2 + g_{23} dx^2 dx^3 + g_{32} dx^3 dx^2 + g_{33} dx^3 dx^3 \\ &= g_{22} d\theta^2 + (g_{23} + g_{32}) d\theta d\phi + g_{33} d\phi^2. \end{aligned} \tag{4.2}$$

We now introduce the complex dyad $q_A(A = \theta, \phi) = (1, i \sin\theta)$ where $q^A = (1, \frac{j}{\sin\theta})$ and $j = \sqrt{-1}$. q_A and q^A satisfy the following conditions: $q^A q_A = 0$, $q^A \overline{q}_A = 2$, and $q^A = q^{AB} q_B$, with $q^{AB} q_{BC} = \delta^A_C$ and $q_{AB} = \frac{1}{2}(q_A \overline{q}_B + \overline{q}_A q_B)$, where $\overline{q_A}$ and $\overline{q_B}$ are the complex dyad conjugate of q_A and q_B respectively.

We also introduce the complex quantities U, J defined by

$$U = q_A U^A, \tag{4.3}$$

and

$$J = q^A q^B h_{AB}/2. \tag{4.4}$$

For the spherically symmetric case (Schwarzschild space-time), we take $J = 0$ and $U = 0$. J and U are interlinked, and they contain all the dynamic content of the gravitational filed in the linearized regime [10]. Lastly we introduce the complex differential eth operators ð and $\bar{ð}$ (see [41] for full details). The eth (ð) formalism gives a compact and efficient manner of treating vector and tensor fields on the sphere, as well as their covariant derivatives.

We define the operator ð acting on a quantity $\mathcal{V}$ of spin-weight s, as

$$ð\mathcal{V} = -(\sin\theta)^s \left[\frac{\partial}{\partial\theta} + j \csc\theta \frac{\partial}{\partial\phi}\right] (\sin\theta)^{-s} \mathcal{V} \tag{4.5}$$

which has the property of raising the spin-weight by 1, and similarly we define $\bar{ð}$ as

$$\bar{ð}\mathcal{V} = -(\sin\theta)^{-s} \left[\frac{\partial}{\partial\theta} - j \csc\theta \frac{\partial}{\partial\phi}\right] (\sin\theta)^s \mathcal{V}, \tag{4.6}$$

which has the property of lowering the spin-weight by 1.

For a Schwarzschild space-time, we have $J = U = 0$, and usually we can describe this space-time by $\beta = 0$ and $W = -2M$, or by $\beta = \beta_c$(constant) and $W = (e^{2\beta_c} - 1)r - 2M$. For a spherically symmetric space-time, J and U are zero and thus they can be regarded as a measure of the deviation from spherical symmetry, and in addition, they carry the gravitational radiation information.

4.3 Einstein vacuum equations for the Bondi-Sachs metric linearized about Schwarzschild

We regard the Bondi-Sachs metric quantities J, β, U, and w to be small, i.e

$$J, \beta, U, w = \mathcal{O}(\epsilon) \quad \text{where} \quad W = -2M + w \tag{4.7}$$

By linearizing the Einstein vacuum equations $R_{ij} = 0$ (introduced in chapter 1, sec. 1.1.4) for the Bondi-Sachs metric, they decompose into hypersurface equations- R_{11}, $q^A R_{1A}$, $h^{AB} R_{AB}$ for β, U and W; evolution equations $q^A q^B R_{AB}$ for J; and constraints R_{0i} [13].

a) the hypersurface equations are given by:

$$R_{11} : \frac{4}{r}\beta_{,1} = 0, \tag{4.8}$$

$$q^A R_{1A} : \frac{1}{2r}(4\eth\beta - 2r\eth\beta_{,1} + r\bar{\eth}J_{,1} + r^3 U_{,1,1} + 4r^2 U_{,1}) = 0, \tag{4.9}$$

$$h^{AB} R_{AB} : (4 - 2\eth\bar{\eth})\beta + \frac{1}{2}(\bar{\eth}^2 J + \eth^2 \bar{J}) + \frac{1}{2r^2}(r^4 \eth\bar{U} + r^4 \bar{\eth} U)_{,1} - 2w_{,1} = 0. \tag{4.10}$$

Here J, U, β, and w are of spin-weight 2, 1, 0, and 0 respectively.

b) the evolution equations are given by:

$$q^A q^B R_{AB} : -2\eth^2\beta + (r^2\eth U)_{,1} - 2(r - M)J_{,1} - \left(1 - \frac{2M}{r}\right) r^2 J_{,1,1} + 2r(rJ)_{,0,1} = 0. \tag{4.11}$$

c) and the constraint equations are given by:

$$R_{00} : \frac{1}{2r^3}\big(r(r-2M)w_{,1,1} + \eth\bar\eth w + 2(r-2M)\eth\bar\eth\beta - Mr(\eth\overline{U} + \bar\eth U) \\ -4r(r-2M)\beta_{,0} - r^3(\eth\overline{U} + \bar\eth U)_{,0} + 2rw_{,0}\big) = 0, \qquad (4.12)$$

$$R_{01} : \frac{1}{4r^2}\big(2rw_{,1,1} + 4\eth\bar\eth\beta - (r^2\eth\overline{U} + r^2\bar\eth U_{,1}\big) = 0, \qquad (4.13)$$

$$q^A R_{0A} : \frac{1}{4r^2}\big(2r\eth w_{,1} - 2\eth w + 2r^2(r-2M)(4U_{,1} + rU_{,1,1}) + 4r^2 U \\ r^2(\eth\bar\eth U - \eth^2\overline{U}) + 2r^2\bar\eth J_{,0} - 2r^4 U_{,0,1} - 4r^2\eth\beta_{,0}\big) = 0. \qquad (4.14)$$

4.3.1 Complex notation

At this stage we must deal with a notational issue concerning the use of complex numbers to represent physical quantities. J and U are complex and are used as a convenient representation of metric quantities with two real components. However, it is also common practice to represent oscillations in time as $e^{i\sigma u}$. More precisely, it is common to write

$$\begin{aligned} A_R\cos\sigma u - A_{Im}\sin\sigma u &= Re\{(A_R + iA_{Im})e^{i\sigma u}\} \\ &= Re\{Ae^{i\sigma u}\}, \end{aligned} \qquad (4.15)$$

with $A = A_R + iA_{Im}$.

Not only is the above a more compact notation, but also it is much easier to manipulate $e^{i\sigma u}$ (by means of differential and integral operators) than sine or cosine function.

The difficulty is that the complex nature of J and U on the one hand, and of $e^{i\sigma u}$ on the other, have no connection with each other. The simplest way around the problem is to keep complex representations for both $e^{i\sigma u}$, as well as J and U, by using i in $e^{i\sigma u}$ with $i^2 = -1$, and j in J and U with $j^2 = -1$, but $i \neq j$ and ij not simplifiable. Although this construction appears similar to quaternion theory, it is, in fact, different. A new algebra has not been constructed, and only addition and multiplication will be performed. In general, an inverse may not exist, so division is not permitted.

The above construction was not made in [13] because in that work it was possible to neglect the imaginary component in J and U. However, as we shall see in Eqs. (5.72) and (5.73), that is not the case here.

4.3.2 Separation of variables

We start by making the assumption that J, U, β, and w can be written as [13]

$$J = J_0(r)e^{i\sigma u}\eth^2 Y_{l0}, \;\; U = U_0(r)e^{i\sigma u}\eth Y_{l0}, \;\; \beta = \beta_0(r)e^{i\sigma u}Y_{l0}, \quad w = w_0(r)e^{i\sigma u}Y_{l0}, \tag{4.16}$$

where l, r_0, and σ are fixed, and where for example,

$$J_0(r) = J_{0r}(r) + iJ_{0i}(r) + jJ_{0j} + ijJ_{0ij}(r), \tag{4.17}$$

$$U_0(r) = U_{0r}(r) + iU_{0i}(r) + jU_{0j} + ijU_{0ij}(r), \tag{4.18}$$

$$\beta_0(r) = \beta_{0r}(r) + i\beta_{0i}(r), \tag{4.19}$$

and

$$w_0(r) = w_{0r}(r) + iw_{0i}(r). \tag{4.20}$$

Note also that we are using $m = 0$ in Y_{lm}, because only that case shall be needed later, and it avoids having to introduce the Z_{lm} as in [13], $J_0(r)$ represent g_{22}, g_{33} and g_{23}, while $U_0(r)$ represents g_{02} and g_{03}, β_0 is independent of r, and w_0 is a function of r. The effect of $\eth$ and $\eth^2$ acting on spherical harmonics Y_{l0} is

$$\eth Y_{l0} = \sqrt{-L_2}_1 Y_{l0} \quad \text{and} \quad \eth^2 Y_{l0} = \sqrt{-(l-1)L_2(l+2)}_2 Y_{l0}, \tag{4.21}$$

where

$$L_2 = -l(l+1). \tag{4.22}$$

We use the eigenfunction decomposition technique to reduce the evolution equations (4.8), (4.9), (4.10) and the constraint equations (4.11) to ordinary differential equations using the ansatz Eq. (4.16) and Eqs. (4.17) to (4.20) to get [13]

$$\frac{4}{r}\beta_{0r,1}e^{i\sigma u}Y_{l0} = 0, \tag{4.23}$$

$$\frac{1}{2r}(4\beta_{0r} - 2r\beta_{0r,1} + r^3 U_{0r,1,1} + 4r^2 U_{0r,1} + (2+L_2)rJ_{0r})e^{i\sigma u}Y_{l0} = 0, \tag{4.24}$$

$$\left(2(2-L_2)\beta_{0r} + L_2(L_2+2)J_{0r} + \frac{1}{R^2}(R^4 L_2 U_{0r})_{,1} - 2\omega_{0r,1}\right)e^{i\sigma u}Y_{l0} = 0, \tag{4.25}$$

$$\left(-2\beta_{0r} + 2U_{0r}r + r^2 U_{0r,1} - 2(r-M)J_{0r,1} - r^2\left(1 - \frac{2M}{r}\right)J_{0r,1,1} + 2ri\sigma(J_{0r} + rJ_{0r,1})\right) \times e^{i\sigma u}\eth^2 Y_{l0} = 0, \tag{4.26}$$

If we continue to solve the above ordinary differential equations (see [13] for the procedure) and taking $l = 2$, we end up with linearized solution of Einstein vacuum equations for Bondi-Sachs metric given by

$$-2J_2(2x + 8Mx^2 + i\sigma) + 2\frac{dJ_2}{dx}(2x^2 + i\sigma x - 7x^3 M) + x^3(1 - 2xM)\frac{d^2J_2}{dx^2} = 0, \tag{4.27}$$

where $J_2(x) \equiv d^2J/dx^2$ and $x = 1/r$.

In the next chapter on the transformation of linear perturbations of a Schwarzschild black hole, we are going to demonstrate heuristically the relation between Eq. (4.27) and the Zerilli equation that we derived in chapter 3, sec. 3.3 for $l = 2$.

4.3.3 The linearized Bondi-Sachs metric when angular momentum is 2

The results of this chapter are going to be used in the next chapter to find J, U, β, and w from the transformed linear perturbations of a Schwarzschild black hole in the case $l = 2$.

The Bondi-Sachs metric (see Eq. (4.1) sec. 4.2) linearized about Schwarzschild background has the following metric components

$$\begin{aligned}
g_{00} &= -1 + \frac{2M}{r} - 2\beta + 4M\beta - \frac{w}{r} && (4.28)\\
g_{01} &= g_{10} = -1 - 2\beta && (4.29)\\
g_{02} &= g_{20} = -r^2 U^\theta && (4.30)\\
g_{03} &= g_{30} = -r^2 \sin^2\theta U^\phi && (4.31)\\
g_{11} &= g_{12} = g_{13} = g_{21} = g_{31} = 0 && (4.32)\\
g_{23} &= g_{32} = r^2 b = r^2 h_{23} = r^2 h_{32} && (4.33)\\
g_{22} &= r^2(1+a) = r^2 h_{22} && (4.34)\\
g_{33} &= r^2(1-a)\sin^2\theta = r^2 h_{33} && (4.35)
\end{aligned}$$

where a and b are functions of r and θ only, and metric quantities β, w, U^θ and U^ϕ are all small. We write β, U, J and w explicitly as:

$$\beta = \beta_0 e^{i\sigma u} Y_{20}(\theta). \tag{4.36}$$

From Eq. (4.3) we have

$$\begin{aligned}
U &= U^\theta + j\frac{U^\phi}{\sin\theta} = -\frac{1}{r^2}\left[g_{02} + j\frac{g_{03}}{\sin\theta}\right]\\
&= U_0(r) e^{i\sigma u}\sqrt{6}\,{}_1Y_{20}(\theta).
\end{aligned} \tag{4.37}$$

From Eq. (4.4) we have

$$\begin{aligned}
J &= \frac{1}{r^2}\frac{q^A q^B g_{AB}}{2}\\
&= a + j\frac{b}{\sin\theta}\\
&= J_0(r) e^{i\sigma u} 2\sqrt{6}\,{}_2Y_{20}(\theta).
\end{aligned} \tag{4.38}$$

Lastly

$$w = w_0(r) e^{i\sigma u} Y_{20}(\theta). \tag{4.39}$$

The spherical harmonics $Y_{20}(\theta)$, ${}_2Y_{20}(\theta)$, and ${}_1Y_{20}(\theta)$ are respectively given by

$$Y_{20}(\theta) = \left(\frac{5^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}}\right)(2 - 3\sin^2\theta), \tag{4.40}$$

$$
{}_2Y_{20}(\theta) = \left(\frac{5^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}}\right) 3(1 - \cos 2\theta), \tag{4.41}
$$

$$
{}_1Y_{20}(\theta) = -\left(\frac{5^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}}\right) 3 \sin 2\theta. \tag{4.42}
$$

4.4 Conclusion

In this chapter, we introduced the Bondi-Sachs metric concepts relevant for our work. We then introduced the linearized Einstein vacuum equations and we reduced them to ordinary differential equations and we solved them to obtain a linearized solution which is needed for the next chapter. We also linearized the Bondi-Sachs metric when $l = 2$ to obtain the expressions for the Bondi-Sachs metric functions w, β, U, and J which we shall use in the next chapter. In the next chapter we are going to transform even and odd-parity perturbations of a Schwarzschild black hole introduced in chapter 3 to the Bondi-Sachs form.

Chapter 5

Transformation of Schwarzschild linear perturbations

5.1 Introduction

In this chapter we illustrate heuristically the relation between the Zerilli equation and Eq. (4.27) since they describe the same linear perturbations of a Schwarzschild black hole. In doing this, we make use of the complex characteristic-frequency belonging to quasi-normal modes (when l=2) of Zerilli's potential to study the convergence of solution series of Eq. (4.27). We then transform odd and even-parity metric perturbations of a Schwarzschild black hole to Bondi-Sachs form. And lastly we compare the transformed metric perturbations with Bondi-Sachs metric (introduced in chap. 4, sec. 4.3.3) to obtain the metric functions J, U, β, and w, where J and U carries the gravitational radiation information.

5.2 The relation between the Zerilli equation and Eq.(4.27)

In illustrating the relation between the Zerilli equation and Eq. (4.27), we want to see that if $\sigma = 0.74734 + 0.17792i$, then $\exists\ J_2(x)$(not everywhere zero) such that

$$-2J_2(2x + 8Mx^2 + i\sigma) + 2\frac{dJ_2}{dx}(2x^2 + i\sigma x - 7x^3M) + x^3(1 - 2xM)\frac{d^2J_2}{dx^2} = 0 \quad (5.1)$$

has a solution with

$$J_2(x = 0) = 0 \quad \text{and} \quad J_2(x = \frac{1}{2}) = 0, \quad (5.2)$$

where $0.74734 + 0.17792i$ is the complex characteristic-frequency for $l = 2$ belonging to the quasi-normal mode of Zerilli's potential (see Table 3.2).

Illustration. We want to illustrate that $\sigma = 0.74734 + 0.17792i$ satisfies Eqs. (5.1) and (5.2). Bishop (2005) [13] obtained the series solutions of Eq. (5.1) about $x = 0$ and $x = \frac{1}{2}$. In investigating the convergence, we constructed two Maple programs: *ti.map* and *th.map* (see Appendix E), that produce 15 order terms for the two series. Program *th.map* works out J as a series in y, where $y = r - 2$, written as J_y here and as $J(6)$ in the program. The program *ti.map* works out J as a series in x, where $x = 1/r$, written as J_x here and as *jout* in the program. One must note that *om* in both programs is actually σ.

For the program *ti.map* we have

$$\begin{aligned}
J_x &= \frac{1}{6}x^3 + \frac{\frac{3}{8}Mx^5}{\sigma} - \frac{7}{12}\frac{Mx^6}{\sigma^2} + \frac{5}{16}M(M5\sigma + 4)x^7\frac{1}{\sigma^3} - \frac{3}{32}M(M77\sigma + 36)x^8\frac{1}{\sigma} \\
&+ \frac{35}{1152}M(M^2 315\sigma^2 + 1022M\sigma + 360)x^9\frac{1}{\sigma^5} \\
&- \frac{1}{96}M(8393M^2\sigma^2 + 13546M\sigma + 3960)x^{10}\frac{1}{\sigma^6} \\
&+ \frac{63}{2816}M(M^3 3465\sigma^3 + 28028M^2\sigma^2 + 31052M\sigma + 7920)x^{11}\frac{1}{\sigma^7} \\
&- \frac{5}{4224}M(986755M^3\sigma^3 + 3634228M^2\sigma^2 + 3142532M\sigma + 720720)x^{12}\frac{1}{\sigma^8} \\
&+ \frac{21}{13312}M(495495M^4\sigma^4 + 7955024M^3\sigma^3 + 18977348M^2\sigma^2
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 13702688M\sigma + 2882880)x^{13}\frac{1}{\sigma^9} \\
&- \quad \frac{15}{73216}M(85449595M^4\sigma^4 + 590566672M^3\sigma^3 \\
&+ \quad 1055102708M^2\sigma^2 + 662747040M\sigma + 129729600)x^{14}\frac{1}{\sigma^{10}}
\end{aligned} \tag{5.3}
$$

For the program *th.map* we have written the series up to the seventh term, this is due to the fact that from the eighth term the series becomes long and complicated.

$$
\begin{aligned}
J_y \quad = \quad & -\frac{1}{8}\frac{y^2}{M^2} + \frac{1}{24}\frac{(-3+14M\sigma)y^3}{M^3(-3+4M\sigma)} - \frac{1}{128}\frac{(64M^2\sigma^2 - 37M\sigma + 8)y^4}{M^4(-3+4M\sigma)(-1+M\sigma)} \\
+ \quad & \frac{1}{64}\frac{(54M\sigma - 105M^2\sigma^2 + 96M^3\sigma^3 - 10)y^5}{M^5(-3+4M\sigma)(-1+M\sigma)(-5+4M\sigma)} \\
- \quad & \frac{1}{1536}\frac{(360 - 2184M\sigma + 5069M^2\sigma^2 - 5640M^3\sigma^3 + 3200M^4\sigma^4)y^6}{M^6(-3+4M\sigma)(-1+M\sigma)(-5+4M\sigma)(-3+2M\sigma)} \\
+ \quad & \frac{1}{512}\frac{(-7383M^2\sigma^2 + 9986M^3\sigma^3 - 7280M^4\sigma^4 + 2816M^5\sigma^5 - 420 + 2790M\sigma)y^7}{M^7(-3+4M\sigma)(-1+M\sigma)(-5+4M\sigma)(-3+2M\sigma)(-7+4M\sigma)} \\
& \ldots
\end{aligned} \tag{5.4}
$$

We now substitute $\sigma = 0.74734 + 0.17792i$ and $M = 1$, in Eqs. (5.3) and (5.4) to obtain the following series

$$
\begin{aligned}
J_x \quad = \quad & \frac{1}{6}x^3 + (0.47 - 0.11i)x^5 - (0.88 - 0.45i)x^6 + (0.45\times10^1 - 0.30\times10^1 i)x^7 \\
- \quad & (0.18\times10^2 - 0.18\times10^2 i)x^8 + (0.85\times10^2 - 0.12\times10^3)x^9 \\
- \quad & (0.39\times10^3 - 0.89\times10^3 i)x^{10} + (0.15\times10^4 - 0.68\times10^4 i)x^{11} \\
- \quad & (0.16\times10^4 - 0.55\times10^5 i)x^{12} - (0.78\times10^5 + 0.46\times10^6 i)x^{13} \\
+ \quad & (0.16\times10^7 + 0.41\times10^7 i)x^{14},
\end{aligned} \tag{5.5}
$$

and

$$
\begin{aligned}
J_y \quad = \quad & -\frac{1}{8}y^2 + (0.14 - 0.44i)y^3 + (0.59 - 0.20i)y^4 + (0.41 + 0.10i)y^5 \\
+ \quad & (0.16 + 0.13i)y^6 + (0.03 + 0.07i)y^7 + (0.03\times10^{-1} + 0.02i)y^8
\end{aligned}
$$

$$
\begin{aligned}
&- \quad (0.17 \times 10^{-2} - 0.48 \times 10^{-2} i) y^9 - (0.40 \times 10^{-3} - 0.90 \times 10^{-3} i) y^{10} \\
&- \quad (0.24 \times 10^{-3} - 0.93 \times 10^{-3} i) y^{11} + (0.18 \times 10^{-4} + 0.20 \times 10^{-4} i) y^{12} \\
&- \quad (0.28 \times 10^{-4} + 0.49 \times 10^{-5} i) y^{13} + (0.11 \times 10^{-4} + 0.23 \times 10^{-5}) y^{14} \\
&- \quad (0.58 \times 10^{-5} + 0.13 \times 10^{-5}) y^{15}.
\end{aligned}
\tag{5.6}
$$

Data obtained from Eqs. (5.5) and (5.6) is represented in the following Tables

	N	11	12	13	14
x					
0.01		1.67×10^{-5}	1.67×10^{-5}	1.67×10^{-5}	1.67×10^{-5}
0.1		1.71×10^{-4}	1.71×10^{-4}	1.71×10^{-4}	1.71×10^{-4}
0.15		5.93×10^{-4}	5.94×10^{-4}	5.94×10^{-4}	5.94×10^{-4}
0.2		1.44×10^{-4}	1.44×10^{-4}	1.44×10^{-4}	1.44×10^{-4}
0.21		1.68×10^{-3}	1.74×10^{-3}	1.73×10^{-3}	1.67×10^{-3}
0.22		1.93×10^{-3}	2.04×10^{-3}	2.04×10^{-3}	2.00×10^{-3}
0.23		2.21×10^{-3}	2.40×10^{-3}	2.43×10^{-3}	2.51×10^{-3}
0.24		3.00×10^{-3}	3.00×10^{-3}	3.00×10^{-3}	3.44×10^{-3}
0.25		2.84×10^{-3}	3.38×10^{-3}	3.74×10^{-3}	5.17×10^{-3}
0.5		7.12×10^{-1}	2.69	10.76	46.88
1		8.40×10^{2}	6.13×10^{3}	4.89×10^{4}	4.23×10^{5}

Table 5.1: $|J_x|$ approximated by $\sum_3^N a_n x^n$ for x, N shown, $M = 1$, $\sigma = 0.74734 + 0.17792i$.

	N	12	13	14	15
y					
0.01		1.14×10^{-3}	1.14×10^{-3}	1.14×10^{-3}	1.14×10^{-3}
0.15		1.14×10^{-3}	1.14×10^{-3}	1.14×10^{-3}	1.14×10^{-3}
0.2		4.72×10^{-3}	4.72×10^{-3}	4.72×10^{-3}	4.72×10^{-3}
0.5		7.29×10^{-3}	7.29×10^{-3}	7.29×10^{-3}	7.29×10^{-3}
1		1.25	1.25	1.25	1.25
1.1		1.93	1.93	1.93	1.93
1.2		2.91	2.91	2.91	2.90
1.5		8.79	8.78	8.79	8.78
2		42.77	42.56	42.73	42.55

Table 5.2: $|J_y|$ approximated by $\sum_3^N a_n y^n$ for y, N shown, $M = 1$, $\sigma = 0.74734 + 0.17792i$.

We were not able to make a rigorous statement about the convergence of Eqs. (5.5) and (5.6), and so instead make heuristic remarks. It appears that both series converge for small x, y, that J_y has a radius of convergence of about $y = 1.1$, and that J_x has a radius of convergence of about $x = 0.2$, which corresponds to $y = 3$. Thus there is no overlap region in which both series are valid, so it was not possible to determine whether $\sigma = 0.74734 + 0.17792i$ causes Eqs. (5.1) and (5.2) to be satisfied.

5.3 Transformation of the even-parity metric perturbations to Bondi-Sachs form

5.3.1 Transformation procedure

The unperturbed Schwarzschild metric in spherical coordinates (t, r, θ, ϕ) is given by Eq. (2.40) and the perturbed Schwarzschild metric up to the first order in ε is included in Appendix C together with the computer programm that calculates it.

We start by transforming t to u by performing the following transformation

$$u = t - F(r) - \varepsilon f(r) e^{i\sigma t} P_2(\theta) \tag{5.7}$$

where $F(r)$ and $f(r)$ are functions that needs to be determined and $P_2(\theta) = \left[\frac{3}{2}\cos^2\theta - \frac{1}{2}\right]$ are the Legendre functions. Differentiating Eq. (5.7) we get

$$\begin{aligned} dt = du + \varepsilon f(r) i\sigma e^{i\sigma t} P_2(\theta) du + F(r)_{,r} dr + \varepsilon F(r)_{,r} f(r) i\sigma e^{i\sigma t} P_2(\theta) dr \\ + \varepsilon f(r)_{,r} e^{i\sigma t} P_2(\theta) dr + \varepsilon f(r) e^{i\sigma t} P_2(\theta)_{,\theta} d\theta. \end{aligned} \tag{5.8}$$

Then we substitute Eq. (5.8) into the perturbed metric and we chose a function $F(r)$ such that the transformed metric after the substitution of Eq. (5.8) has the coefficient of dr^2 zero to the zeroth order in ε. Similarly we chose a function $f(r)$ such that the coefficient of dr^2 is zero to 1st order in ε. The transformed metric is given by line b17[1] in Appendix C. We found functions $F(r)$ and $f(r)$ to be

$$F(r) = r + 2Mln(r - 2M), \tag{5.9}$$

and

$$f(r) = e^{(\frac{ri\sigma}{r-2M})} \int \frac{e^{(-\frac{ri\sigma}{-r+2M})}(rN(r) - rL(r))}{-r + 2M} dr. \tag{5.10}$$

After the above transformation, we note that $e^{i\sigma t}$ now has the form $e^{i\sigma u + i\sigma F(r)}$. We also note that from the transformed metric (line b17) there is a $drd\theta$ term that needs to be removed. We remove this term by transforming θ to ψ by performing the following transformation

$$\theta \rightarrow \psi = \theta - \varepsilon\alpha(u, r, \psi) \tag{5.11}$$

where $\alpha(u, r, \psi)$ (ψ not be confused with that of Regge-Wheeler equation in chapter 3, sec. 3.3) is a function that needs to be determined. We then differentiate Eq. (5.11) to get

$$d\theta = d\psi + \varepsilon\alpha(u, r, \psi)_{,u} du + \varepsilon\alpha(u, r, \psi)_{,r} dr + \varepsilon\alpha(u, r, \psi)_{,\psi} d\psi \tag{5.12}$$

We substitute Eq. (5.12) into the transformed metric and apply the condition that the coefficient of $drd\psi$ must be zero to 1st order in ε. We then work out the complete transformed metric up to 1st order in ε and transform $\sin^2\theta$ as follows

$$\begin{aligned} \sin^2\theta \quad &\rightarrow \quad \sin^2(\psi - \varepsilon\alpha(u, r, \psi)) \\ &= \quad (\sin\psi - \varepsilon\alpha(u, r, \psi)\cos\psi)^2 \\ &= \quad \sin^2\psi(1 - 2\varepsilon\alpha(u, r, \psi)\cot\psi). \end{aligned} \tag{5.13}$$

We found $\alpha(u, r, \psi)$ to be

$$\alpha(u, r, \psi) = -3\sin\psi\cos\psi e^{i\sigma u} I_f(r), \tag{5.14}$$

where

$$I_f(r) = \int \frac{e^{i\sigma F(r)} f(r)}{r^2} dr. \tag{5.15}$$

[1]due to the long and complexity nature of the transformed metrics of this section, we opted not to include them in this section, but we shall simply refer to them in Appendix C.

Finally, we transform r to a new r' by performing the following transformation

$$r \rightarrow r' = r + \varepsilon\Omega(u, r, \psi) \tag{5.16}$$

were $\Omega(u, r, \psi)$ is a function that needs to be determined. Eq. (5.16) satisfy the following condition

$$r'^4 \sin^2\psi = g_{22} \cdot g_{33}. \tag{5.17}$$

We use Eq. (5.17) to find $\Omega(u, r, \psi)$, and it was found to be

$$\Omega(u, r, \psi) = -\frac{1}{4} e^{i\sigma u}(3\cos^2\psi - 1) r \left[3 I_f(r) + e^{i\sigma F(r)}[3V(r) - T(r)]\right], \tag{5.18}$$

The complete transformed metric is given by line b93.

5.3.2 The transformed metric

After the above transformation, we found the transformed metric to be given by

$$\begin{aligned}
g_{00} &= -\frac{1}{r^2}\left[-2Mr - 2\varepsilon\left[\frac{\partial}{\partial u}\Omega(u, r, \psi)\right] r^2 - 2M\varepsilon\Omega(u, r, \psi) + r^2\right. \\
&\quad + 2e^{(i\sigma F(r) + i\sigma u)}\varepsilon f(r) i\sigma P_2(\psi) r^2 - 4e^{(i\sigma F(r) + i\sigma u)}\varepsilon f(r) i\sigma P_2(\psi) Mr \\
&\quad \left. - 4e^{(i\sigma F(r) + i\sigma u)}\varepsilon P_2(\psi) N(r) Mr + 2e^{(i\sigma F(r) + i\sigma u)}\varepsilon P_2(\psi) N(r) r^2\right], \qquad (5.19) \\
g_{01} &= g_{10} = -1 + \varepsilon\left[\frac{\partial}{\partial r}\Omega(u, r, \psi)\right] - 2e^{(i\sigma F(r) + i\sigma u)}\varepsilon P_2(\psi) N(r) \\
&- 2e^{(i\sigma F(r) + i\sigma u)}\varepsilon f(r) i\sigma P_2(\psi), \qquad (5.20) \\
g_{02} &= g_{20} = \frac{\varepsilon}{r}\left[\left[\frac{\partial}{\partial \psi}\Omega(u, r, \psi)\right] r + 2e^{(i\sigma F(r) + i\sigma u)} f(r)\left[\frac{\partial}{\partial \psi} P_2(\psi)\right] M\right. \\
&\quad \left. + \left[\frac{\partial}{\partial u}\Omega(u, r, \psi)\right] r^3 - e^{(i\sigma F(r) + i\sigma u)} f(r)\left[\frac{\partial}{\partial \psi} P_2(\psi)\right] r\right], \qquad (5.21) \\
g_{22} &= r\left[-2\varepsilon\Omega(u, r, \psi) + r - 12e^{(i\sigma F(r) + i\sigma u)} r\varepsilon V(r)\cos^2\psi + 2e^{(i\sigma F(r) + i\sigma u)} r\varepsilon P_2(\psi) T(r)\right. \\
&\quad \left. 6e^{(i\sigma F(r) + i\sigma u)} r\varepsilon V(r) + 2r\varepsilon\left[\frac{\partial}{\partial \psi}\Omega(u, r, \psi)\right]\right], \qquad (5.22) \\
g_{33} &= -r\left[2\varepsilon\Omega(u, r, \psi) + r\cos^2\psi - r + 2\sin\psi r\varepsilon\alpha\cos\psi - 2\cos^2\psi\varepsilon\Omega(u, r, \psi)\right. \\
&\quad + 6e^{(i\sigma F(r) + i\sigma u)} r\varepsilon V(r)\cos^2\psi - 6e^{(i\sigma F(r) + i\sigma u)} r\varepsilon\cos^4\psi V(r) \\
&\quad \left. - 2e^{(i\sigma F(r) + i\sigma u)} r\varepsilon P_2(\psi) T(r) + 2e^{(i\sigma F(r) + i\sigma u)} r\varepsilon P_2(\psi) T(r)\cos^2\psi\right]. \qquad (5.23)
\end{aligned}$$

which simplifies to

$$g_{00} = -1 + \frac{2M}{r} + 2\varepsilon e^{i\sigma u} P_2(\psi) e^{i\sigma F(r)}\left[\frac{2M}{r} - 1\right][f(r) i\sigma + N(r)]$$

$$
\begin{aligned}
&+\ \frac{1}{r}\left[2\varepsilon\left[\frac{\partial}{\partial u}\Omega(u,r,\psi)\right]r+\frac{2M}{r}\varepsilon\Omega(u,r,\psi)\right], && (5.24)\\
g_{01} &= g_{10}=-1-\left[\varepsilon\left[\frac{\partial}{\partial r}\Omega(u,r,\psi)\right]+2e^{i\sigma u}\varepsilon P_2(\psi)e^{i\sigma F(r)}[f(r)i\sigma+N(r)]\right],\\
g_{02} &= g_{20}=-r^2\left[-\varepsilon\left[\frac{\partial}{\partial_\psi}\Omega(u,r,\psi)\right]\frac{1}{r^2}-2\varepsilon e^{i\sigma u}\left[\frac{\partial}{\partial_\psi}P_2(\psi)\right]e^{i\sigma F(r)}f(r)\frac{M}{r^3}\right.\\
&\left.-\varepsilon\left[\frac{\partial}{\partial u}\alpha(u,r,\psi)\right]+\varepsilon e^{i\sigma u}\left[\frac{\partial}{\partial\psi}P_2(\psi)\right]e^{i\sigma F(r)}f(r)\frac{1}{r^2}\right], && (5.25)\\
g_{22} &= r^2\left[1+\left[-\frac{2\varepsilon}{r}\Omega(u,r,\psi)+2e^{i\sigma u}\varepsilon P_{2(\psi)}e^{i\sigma F(r)}T(r)+6e^{i\sigma u}\varepsilon[1-2\cos^2\psi]e^{i\sigma F(r)}V(r)\right.\right.\\
&\left.\left.+2\varepsilon\left[\frac{\partial}{\partial\psi}\alpha(u,r,\psi)\right]\right]\right], && (5.26)\\
g_{33} &= r^2\left[1-\left[\frac{2\varepsilon}{r}\Omega(u,r,\psi)+6e^{i\sigma u}\cos^2\psi\varepsilon e^{i\sigma F(r)}V(r)-2e^{i\sigma u}\varepsilon P_2(\psi)e^{i\sigma F(r)}T(r)\right.\right.\\
&\left.\left.-6\varepsilon\cos^2\psi e^{i\sigma u}I_f(r)\right]\right]\sin^2\psi, && (5.27)\\
& && (5.28)
\end{aligned}
$$

$F(r)$, $f(r)$, $\alpha(r)$, $\Omega(u,r,\psi)$ are functions given by Eqs. (5.9), (5.10), (5.14), and (5.18) respectively.

5.3.3 Comparison of the transformed metric with the linearized Bondi-Sachs metric

By comparing the transformed even-parity metric perturbations with the linearized Bondi-Sachs metric (chapter 4, sec. 4.3.3) and noticing that $g_{11}=g_{12}=g_{13}=g_{21}=g_{31}=g_{23}=g_{32}=g_{03}=g_{30}=0$, we found that β, U, J, w are given by

$$w=-\left[2\varepsilon\left[\frac{\partial}{\partial u}\Omega(u,r,\psi)\right]r+\frac{2M}{r}\varepsilon\Omega(u,r,\psi)\right], \tag{5.29}$$

$$\beta=\left[\varepsilon\left[\frac{\partial}{\partial r}\Omega(u,r,\psi)\right]+2e^{i\sigma u}\varepsilon P_2(\psi)e^{i\sigma F(r)}[f(r)i\sigma+N(r)]\right], \tag{5.30}$$

$$
\begin{aligned}
U &= 2\left[-\varepsilon\left[\frac{\partial}{\partial_\psi}\Omega(u,r,\psi)\right]\frac{1}{r^2}-2\varepsilon e^{i\sigma u}\left[\frac{\partial}{\partial_\psi}P_2(\psi)\right]e^{i\sigma F(r)}f(r)\frac{M}{r^3}\right.\\
&\left.-\varepsilon\left[\frac{\partial}{\partial u}\alpha(u,r,\psi)\right]+\varepsilon e^{i\sigma u}\left[\frac{\partial}{\partial\psi}P_2(\psi)\right]e^{i\sigma F(r)}f(r)\frac{1}{r^2}\right],
\end{aligned}
\tag{5.31}
$$

$$
\begin{aligned}
J &= -\frac{2\varepsilon}{r}\Omega(u,r,\psi) + 2e^{i\sigma u}\varepsilon P_{2(\psi)}e^{i\sigma F(r)}T(r) + 6e^{i\sigma u}\varepsilon[1-2\cos^2\psi]e^{i\sigma F(r)}V(r) \\
&+ 2\varepsilon\left[\frac{\partial}{\partial\psi}\alpha(u,r,\psi)\right],
\end{aligned}
\tag{5.32}
$$

or

$$
\begin{aligned}
J &= \frac{2\varepsilon}{r}\Omega(u,r,\psi) + 6e^{i\sigma u}\cos^2\psi\varepsilon e^{i\sigma F(r)}V(r) - 2e^{i\sigma u}\varepsilon P_2(\psi)e^{i\sigma F(r)}T(r) \\
&- 6\varepsilon\cos^2\psi e^{i\sigma u}I_f(r).
\end{aligned}
\tag{5.33}
$$

By substituting functions (5.9), (5.10), (5.14), and (5.18) into Eqs. (5.29), (5.30), (5.31), and (5.32), ω, β, U, and J simplify to

$$
w = w_0(r)\varepsilon e^{i\sigma u}Y_{20}(\theta), \tag{5.34}
$$

where

$$
w_0(r) = \left(\frac{4\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right)\frac{1}{2}r\left[i\sigma r + \frac{M}{r}\right]\left[3I_f(r) + e^{i\sigma F(r)}[3V(r) - T(r)]\right], \tag{5.35}
$$

$$
\beta = \beta_0(r)\varepsilon e^{i\sigma u}Y_{20}(\theta), \tag{5.36}
$$

where

$$
\begin{aligned}
\beta_0(r) &= -\left(\frac{4\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right)\left[\frac{1}{4}\left\{3[I_f(r) + rI'_f(r)] + e^{i\sigma F(r)}[3[(1 - ri\sigma F'(r))V(r) + rV'(r)]\right.\right. \\
&\left.\left. -[(1 - ri\sigma F(r))T(r) + rT'(r)]]\right\} + e^{i\sigma F(r)}[f(r)i\sigma + N(r)]\right]
\end{aligned}
\tag{5.37}
$$

$$
U = \varepsilon e^{i\sigma u}{}_1Y_{20}U_0(r), \tag{5.38}
$$

where

$$
U_0(r) = \frac{1}{2}\left[-\frac{1}{2r}[3I_f(r) + e^{i\sigma F(r)}[3V(r) - T(r)]] + I_f(r) + \frac{e^{i\sigma F(r)}}{r^2}[2M - 1]\right] \tag{5.39}
$$

$$
J = \varepsilon e^{i\sigma u}{}_2Y_{20}J_0(r) \tag{5.40}
$$

where

$$J_0(r) = \left(\frac{4\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right)\frac{1}{2\sqrt{6}}\left\{-\frac{1}{2}\left[\frac{1}{2}[3I_f(r)+e^{i\sigma F(r)}[3V(r)-T(r)]]+e^{i\sigma F(r)}T(r)\right]\right.$$
$$\left.+2[e^{i\sigma F(r)}V(r)+I_f(r)]\right\}, \tag{5.41}$$

We have used the trigonometric identities:

$$\cos a\,\cos b = \frac{1}{2}[\cos(a-b)-\cos(a+b)] \quad \text{and} \quad \sin^2\theta+\cos^2\theta = 1 \tag{5.42}$$

to simplify $\frac{3}{2}\cos^2\psi - \frac{1}{2}$ to $\frac{1}{4}[3(\cos 2\psi - 1)]$ in Eq. (5.32).

5.4 Interpreting the complex quantities

The expressions for β, w, J and U obtained in Sec. 5.3.3 involve the complex quantity i, but not j. Thus, here, the interpretation is straightforward: in all cases β, w, J, and U mean the real part of the given expression.

5.5 Transformation of the odd-parity metric perturbations to Bondi-Sachs form

5.5.1 Transformation procedure

Maple results for this section are included in appendix C. From Eq. (3.2) and by the definition of the odd-parity perturbations which we have defined in chapter 3, sec. 3.3, we have

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta(d\phi - \omega(r,\theta,u)dt$$
$$- q_2(r,\theta,t)dr - q_3(r,\theta,t)d\theta)^2. \tag{5.43}$$

Since $\omega(r,\theta,t)$, $q_2(r,\theta,t)$ and $q_3(r,\theta,t)$ are very small, we then have

$$\omega(r,\theta,t) = \varepsilon\omega(r)3\cos\theta e^{i\sigma t}, \tag{5.44}$$

$$q_2(r,\theta,t) = \varepsilon q_2(r)3\cos\theta e^{i\sigma t}, \tag{5.45}$$

$$q_3(r,\theta,t) = \varepsilon q_3(r)3\sin\theta e^{i\sigma t}. \tag{5.46}$$

We start by transforming t to u by the following transformation

$$u = t - F(r), \tag{5.47}$$

where $F(r)$ is a function that needs to be determined. Differentiating this transformation we obtain

$$dt = du + F'(r)dr. \tag{5.48}$$

By substituting Eq. (5.48) into Eq. (5.43) and choosing the function $F(r)$ such that the transformed metric has the coefficient of dr^2 zero to the zeroth order in ε, and noting that $e^{i\sigma t}$ now has the form $e^{i\sigma(u+F(r))}$; we found the transformed metric to be (see line a8)

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)du^2 - 2\left(1-\frac{2M}{r}\right)\left(\frac{\partial}{\partial r}F(r)\right)dudr + r^2 d\theta^2 \\ &+ r^2\sin\theta^2(d\phi - \varepsilon\omega(r)3\cos\theta e^{i\sigma(u+F(r))}du - (\varepsilon\omega(r)3\cos\theta e^{i\sigma(u+F(r))}\frac{\partial}{\partial r}F(r) \\ &+ \varepsilon q_2(r)3\cos\theta e^{i\sigma(u+F(r))})dr - \varepsilon q_3(r)3\sin\theta e^{i\sigma(u+F(r))}d\theta)^2, \end{aligned} \tag{5.49}$$

where a function $F(r)$ was found to be

$$F(r) = r + 2M\ln(r-2M). \tag{5.50}$$

We then transform ϕ to ψ by performing the following transformation

$$\psi = \phi + g(r,\theta,u) \tag{5.51}$$

where $g(r,\theta,u)$ is a function that needs to be determined. Differentiating this transformation we obtain

$$d\phi = d\psi - \left[\frac{\partial}{\partial r}g(r,\theta,u)\right]dr - \left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right]d\theta - \left[\frac{\partial}{\partial u}g(r,\theta,u)\right]du \tag{5.52}$$

Then by substituting Eq. (5.52) into Eq. (5.49) and choosing $g(r,\theta,u)$ such that the transformed metric after the substitution has the coefficient of dr^2 zero we get (see line a39)

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)du^2 - 2dudr + r^2d\theta^2 + r^2\sin^2\theta\left[d\psi - \left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right]d\theta\right. \\ &\quad - \left[\frac{\partial}{\partial u}g(r,\theta,u)\right]du - \varepsilon\omega(r)3\cos\theta e^{i\sigma(u+F(r))}du \\ &\quad \left.-\varepsilon q_3(r)3\sin\theta e^{i\sigma(u+F(r))}d\theta\right]^2, \end{aligned} \tag{5.53}$$

where a function $g(r,\theta,u)$ was found to be

$$g(r,\theta,u) = -3\varepsilon\cos\theta e^{i\sigma u}\left[I_\omega(r) + I_{q2(r)}\right] \tag{5.54}$$

where

$$I_\omega(r) = \int \frac{e^{i\sigma F(r)}[-\omega(r)r]}{-r+2M}dr \quad \text{and} \quad I_{q_2}(r) = \int e^{i\sigma F(r)}q_2(r)dr. \tag{5.55}$$

5.5.2 The transformed metric

After the above transformation procedure, we found the transformed metric to be

$$\begin{aligned}
g_{00} &= -\ \frac{1}{r}\left[r - 2M - r^3\left[\frac{\partial}{\partial u}g(r,\theta,u)\right]^2 + r^3\left[\frac{\partial}{\partial u}g(r,\theta,u)\right]^2\cos^2\theta\right.\\
&\quad -6r^3\left[\frac{\partial}{\partial u}g(r,\theta,u)\right]\varepsilon\omega(r)\cos\theta e^{i\sigma(u+F(r))}\\
&\quad \left.+6r^3\left[\frac{\partial}{\partial u}g(r,\theta,u)\right]\varepsilon\omega(r)\cos^3\theta e^{i\sigma(u+F(r))}\right], && (5.56)\\
g_{01} &= -1, && (5.57)\\
g_{02} &= -r^2(\cos\theta-1)(\cos\theta+1)\left[3\left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right]\varepsilon\omega(r)\cos\theta e^{i\sigma(u+F(r))}\right.\\
&\quad +\left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right]\left[\frac{\partial}{\partial u}g(r,\theta,u)\right] + 3\sin\theta\left[\frac{\partial}{\partial u}g(r,\theta,u)\right]\times\\
&\quad \left.\varepsilon q_3(r)e^{i\sigma(u+F(r))}\right], && (5.58)\\
g_{03} &= r^2(\cos\theta-1)(\cos\theta+1)\left[3\varepsilon\omega(r)\cos\theta e^{i\sigma(u+F(r))} + \left[\frac{\partial}{\partial u}g(r,\theta,u)\right]\right] && (5.59)\\
g_{23} &= r^2(\cos\theta-1)(\cos\theta+1)\left[\left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right] + 3\varepsilon\sin\theta q_3(r)e^{i\sigma(u+F(r))}\right] && (5.60)\\
g_{22} &= -r^2\left[-1 - \left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right]^2 + \left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right]^2\cos^2\theta\right.\\
&\quad -6\sin\theta\left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right]\varepsilon q_3(r)e^{i\sigma(u+F(r))}\\
&\quad \left.+6\sin\theta\left[\frac{\partial}{\partial\theta}g(r,\theta,u)\right]\varepsilon q_3(r)e^{i\sigma(u+F(r))}\cos^2\theta\right], && (5.61)\\
g_{33} &= -r^2(\cos\theta-1)(\cos\theta+1). && (5.62)
\end{aligned}$$

Substituting Eq. (5.54) in the above metric components, they simplify to

$$g_{00} = -1 + \frac{2M}{r}, \tag{5.63}$$

$$g_{01} = -1, \tag{5.64}$$

$$g_{02} = 0, \tag{5.65}$$

$$g_{03} = -3r^2 \sin^2\theta \varepsilon \cos\theta Re\{e^{i\sigma u}\left[\omega(r)e^{i\sigma F(r)} - i\sigma(I_\omega(r) + I_{q_2}(r))\right]\}, \tag{5.66}$$

$$g_{23} = -3r^2 \sin^3\theta \varepsilon Re\{e^{i\sigma u}[I_\omega(r) + I_{q_2}(r) + q_3(r)e^{i\sigma F(r)}]\}, \tag{5.67}$$

$$g_{22} = r^2, \tag{5.68}$$

$$g_{33} = r^2 \sin^2\theta. \tag{5.69}$$

5.5.3 Comparison of the transformed metric with the linearized Bondi-Sachs metric

By comparing the transformed odd-parity metric perturbations with the linearized Bondi-Sachs metric (chapter 4, sec. 4.3.3) we found that β, U, J, w β_0, U_0, J_0, and w_0 for the transformed odd-parity metric perturbations are given by

$$\beta = 0, \quad \text{and hence} \quad \beta_0(r) = 0. \tag{5.70}$$

$$w = 0, \quad \text{and hence} \quad w_0 = 0. \tag{5.71}$$

From Eq. (4.38) with $a = 0$ we have

$$J = \frac{jb}{\sin\theta}, \tag{5.72}$$

From Eq. (4.37) with $g_{02} = 0$ we have

$$U = -\frac{1}{r^2}\frac{jg_{03}}{\sin\theta}, \tag{5.73}$$

5.6 Interpreting the complex quantities

The expressions for J and U obtained above involve both complex quantities i and j. Taking the real part with respect to i leads to

$$J = -j\varepsilon_2 Y_{20}\left(\frac{4\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right) Re\{e^{i\sigma u}[I_\omega(r) + I_{q_2}(r) + q_3(r)e^{i\sigma F(r)}]\} \tag{5.74}$$

and

$$U = j\varepsilon_1 Y_{20}\left(\frac{4\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right) Re\{e^{i\sigma u}[\omega(r)e^{i\sigma F(r)} - i\sigma(I_\omega(r) + I_{q_2}(r))]\} \tag{5.75}$$

Thus, both U and J are pure imaginary quantities.

5.7 Discussion

We have applied a heuristic method to illustrate the relation between the linearized solution [13] of the linearized Einstein vacuum equations for the Bondi-Sachs metric and the Zerilli equation which describe even-parity perturbations of a Schwarzschild black hole. These two equations describe the same linear perturbations of a Schwarzschild black hole and hence it was necessary to illustrate this relation as Chandrasekhar [25] did between Zerilli and Regge-Wheeler equations, and between Zerilli and Bardeen-Press equations, and between Regge-Wheeler and Bardeen-Press equations in chapter 3. By the heuristic method that we have applied, we were able to illustrate the convergence of both series up to 15 terms. For terms higher than 15, computer programs that we have constructed took too long (i.e. 2 days for 18 terms) to compute the series.

A more mathematically rigorous method can be constructed in such a way that the formulas for the N^{th} term for both series (Eqs. (5.3) and (5.4)) are determined so that their convergence can be studied by using the theory of three-term recurrence relations [40] outlined in chapter 3, sec. 3.8 to determine the conditions under which the sum of the coefficients of these series converges.

The transformation of linear perturbations of a Schwarzschild black hole to Bondi-Sachs is complete. The transformation of even-parity perturbations was much more involved than that of odd-parity perturbations. The end results of the transformation processes for both even and odd-parity perturbations were very different, for example, in the case of odd-parity perturbations, w and β were found to be zero and J and U were found to be purely imaginary and that was not the case for even-parity perturbations were w, β, J and U were found to be real and complicated functions.

All unknown functions; $f(r)$, $F(r)$, $\alpha(u, r, \psi)$, $\Omega(u, r, \psi)$ and $g(r, \theta, u)$ for both even and odd-parity perturbations, were found and verified to be correct by substituting them into the transformed even and odd-parity perturbations, thereby simplifying the transformed perturbations to a point where we were able to find J, U, w, and β. We then wrote J, U, w, and β as spherical harmonics (${}_2Y_{20}$,${}_1Y_{20}$, Y_{20} and Y_{20} respectively) times some functions (J_0, U_0, w_0, and β_0 respectively) times the time dependency factor $e^{i\sigma u}$. Also, for the fact that we were able to extract Y_{20}, ${}_1Y_{20}$, ${}_2Y_{20}$ and J_0, U_0, w_0, β_0 from the transformed odd and even-parity perturbations, meant that the transformation processes were carried out correctly and that all the unknown functions $f(r)$, $F(r)$, $\alpha(u, r, \psi)$, $\Omega(u, r, \psi)$ and $g(r, \theta, u)$ were correctly determined.

Chapter 6

Conclusion

In chapter 1, we started by introducing the following curvature concepts: (a) metric tensor which was the foundation for the rest of the remaining chapters, (b) properties of Riemann curvature tensor, Ricci curvature tensor and Ricci curvature scalar which were needed in chapter 3 for Ricci and Einstein tensors for non-stationary axisymmetric space-times; and (c) vacuum field equations which were important for the work of chapter 4. We also looked at how Einstein vacuum equations are solved using Maple. We introduced the Newman-Penrose formalism and we used it in chapter 2 to express Schwarzschild geometry in this formalism with the aim of deriving Bardeen-Press equation in chapter 3, but firstly, we introduced the Weyl tensor as it was needed in the discussion of this formalism. Lastly we introduced a concise overview of Bondi-Sachs metric with the aim of providing the reader with full general understanding of the dynamics this metric.

In chapter 2 we introduced the concept of spherically symmetric space-times which was important for the rest of this chapter. We derived Schwarzschild solution which was important for chapter 3 were we were defining even and odd-parity perturbations of a Schwarzschild black hole. We stated Birkhorff's theorem to highlight the importance of this theorem in relation to the contents of this dissertation. We also looked at the Schwarzschild geometry and its Carter-Penrose diagrams and causal properties, of which the main aim was to give the reader a geometrical intuition of the structure of a Schwarzschild black hole. Then lastly we described the Schwarzschild geometry in the Newman-Penrose formalism, the aim of which was outlined in the first paragraph of this chapter.

In chapter 3 we computationally recalculated the Ricci tensors for non-stationary axisymmetric space-times and in the process we recovered some common factors and nonlinear terms that did not appear in the original results by Chandrasekhar and we gave R_{00} as an example in the non-linear regime. We noted the fact that these discrepancies do not invalidate the work of Chandrasekhar on linear perturbations of a Schwarzschild black hole. We used these tensors to derive Zerilli and Regge-Wheeler equations. We also proved the relation between the Zerilli and Regge-Wheeler equations via their potentials ($V^{(+)}$ and $V^{(-)}$ respectively) and their one-dimensional wave functions ($Z^{(+)}$ and $Z^{(-)}$ respectively). We derived the Bardeen-Press equation and proved its relation to both the Zerilli and Regge-Wheeler equations. Lastly we introduced quasi-normal modes of a Schwarzschild black hole and we used them in chapter 5, for the purpose of the work of this dissertation, we looked at the work of Leaver [55] and that of Chandrasekhar [24, 25].

In chapter 4 we introduced the Bondi-Sachs metric and in the processes we introduced complex quantities J and U that were important for chapter 5. We then introduced the linearized Einstein vacuum equations for the Bondi-Sachs metric and we solved these equations to obtain a linearized solution which was used in chapter 5 where we were investigating heuristically its relation to the Zerilli equation derived in chapter 3. Lastly, we linearized the Bondi-Sachs metric about a Schwarzschild background so that we could make comparisons with the results obtained in chapter 5.

With the introduction of the above material, we were now in a position to proceed and do the transformation of linear perturbations of a Schwarzschild black hole.

The objective of this dissertation was to extend the results of Chandrasekhar by transforming his results on linear (even and odd-parity) perturbations of a Schwarzschild black hole to the Bondi-Sachs framework and this was achieved in chapter 5. From chapter 5, it appears that the transformation of second order perturbations of a Schwarzschild black hole to Bondi-Sachs form will be extremely difficult to do.

In the future, the extension of the work of this dissertation to a stationary charged

(Reissener-Nordstr$\ddot{o}$m) black hole will be very exciting and hopefully attainable. Similarly, the transformation of linear perturbations(gravitational) of a Kerr black hole will be very exciting to do and it will be a challenging exercise for one to be engaged in. In addition, if we extend the work of this dissertation to Kerr-Newman black hole, we will find it difficult to transform linear perturbations because even and odd-parity perturbations have not yet been decoupled and this is still a challenge to us. Chandrasekhar tried to decouple them and sadly failed to do so. Since then, apart from special cases this problem was not studied at all. Further, while the transformation of Schwarzschild to Bondi-Sachs form is very simple, that of Kerr to Bondi-Sachs form has been obtained only very recently by Bishop *et al.* [14] and is not in an explicit analytic form.

Appendix A

Maple computer programs 1

This program calculates the linearized Ricci tensor of axisymmetric space-times of chapter 3 sec. 3.2. It then checks them against the original versions (linearized) calculated by Chandrasekhar in 1972. In the program $Ricci1[a, b]$ $(a, b = 1..4)$ refers to the computationally computed Ricci tensors while $ChR11$, $ChR12$ etc. refers to the original versions by Chandrasekhar.

A.1 Program: c.map

```
with(linalg):

#Matrix and array declarations gd:=matrix(4,4);        #$g_{ab}$
gu:=matrix(4,4);        #$g^{ab}$ x:=array([ph,r,th,t]);  #$x^a$
gammad:=array(1..4,1..4,1..4);   #$\Gamma_{abc}$
gammau:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
gammaut:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
gammau0:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
gammau1:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
Riemann:=array(1..4,1..4,1..4,1..4);#$R^a_{bcd}$ Ricci:=matrix(4,4);
#$R_{ab}$ Ricci0:=matrix(4,4);      #$R_{ab}$ Ricci1:=matrix(4,4);
```

```
#$R_{ab}$

#Q:=array(1..4,1..4);

#alias(lambda=lambda(r,th,t)); alias(omega=omega(r,th,t));
alias(omegas=omegas(r,th,t)); alias(mu2=m2(r,th,t));
alias(mu3=m3(r,th,t)); alias(nu=nu(r,th,t)); alias(q2=q2(r,th,t));
alias(q3=q3(r,th,t)); alias(qs2=qs2(r,th,t));
alias(qs3=qs3(r,th,t)); alias(nu0=nu0(r)); alias(nu1=nu1(r,t,th));
alias(mu20=mu20(r)); alias(mu21=mu21(r,t,th)); alias(mu30=mu30(r));
alias(mu31=mu31(r,t,th)); alias(lambda0=lambda0(r,th));
alias(lambda1=lambda1(r,t,th)); nu:=nu0+eps*nu1; mu2:=mu20+eps*mu21;
mu3:=mu30+eps*mu31; lambda:=lambda0+eps*lambda1; qs2:=eps*q2;
qs3:=eps*q3; omegas:=eps*omega;
gd[2,2]:=-(exp(2*lambda)*qs2^2+exp(2*mu2)):
gd[2,3]:=-exp(2*lambda)*qs2*qs3:gd[3,2]:=gd[2,3]:
gd[1,2]:=exp(2*lambda)*qs2:gd[2,1]:=gd[1,2]:
gd[2,4]:=-exp(2*lambda)*qs2*omegas:gd[4,2]:=gd[2,4]:
gd[3,3]:=-(exp(2*lambda)*qs3^2+exp(2*mu3)):
gd[1,3]:=exp(2*lambda)*qs3:gd[3,1]:=gd[1,3]:
gd[3,4]:=-exp(2*lambda)*qs3*omegas:gd[4,3]:=gd[3,4]:
gd[1,1]:=-exp(2*lambda):
gd[1,4]:=exp(2*lambda)*omegas:gd[4,1]:=gd[1,4]:
gd[4,4]:=(exp(2*nu)-exp(2*lambda)*omegas^2):

print(gd);

gu:=inverse(gd);

for a1 from 1 to 4 do for b1 from 1 to 4 do for c1 from 1 to 4 do
```

```
  gammad[a1,b1,c1]:=(diff(gd[a1,b1],x[c1]) + diff(gd[c1,a1],x[b1])
                     - diff(gd[b1,c1],x[a1]))/2;
od; od; od;

for a1 from 1 to 4 do for b1 from 1 to 4 do for c1 from 1 to 4 do
  gammaut[a1,b1,c1]:=simplify(sum(gu[a1,d1]*gammad[d1,b1,c1],d1=1..4));
gammau0[a1,b1,c1]:=simplify(subs(eps=0,gammaut[a1,b1,c1]));
gammau1[a1,b1,c1]:=simplify(subs(eps=0,diff(gammaut[a1,b1,c1],eps)));
gammau[a1,b1,c1]:=gammau0[a1,b1,c1]+gammau1[a1,b1,c1]*eps; od; od;
od;

for a1 from 1 to 4 do for b1 from 1 to 4 do
   for c1 from 1 to 4 do for d1 from 1 to 4 do
Riemann[a1,b1,c1,d1]:= diff(gammau[a1,b1,d1],x[c1]) -
diff(gammau[a1,b1,c1],x[d1])+
sum(gammau[a1,e1,c1]*gammau[e1,b1,d1]-gammau[a1,e1,d1]*gammau[e1,b1,c1],e1=1..4);
od; od; od; od;

for a1 from 1 to 4 do for b1 from 1 to 4 do
Ricci[a1,b1]:=simplify(sum(Riemann[e1,a1,e1,b1],e1=1..4));
Ricci0[a1,b1]:=simplify(subs(eps=0,Ricci[a1,b1]));
Ricci1[a1,b1]:=factor(simplify(subs(eps=0,diff(Ricci[a1,b1],eps))));
od; od;

r1:={diff(q2,t)=Q20+diff(omega,r), diff(q3,t)=Q30+diff(omega,th),
diff(q2,th)=Q23+diff(q3,r)};

r2:={diff(q2,r,t)=Q20_2+diff(omega,r,r),
     diff(q3,t,th)=Q30_3+diff(omega,th,th),
     diff(q2,th,th)=Q23_3+diff(q3,th,r),
```

```
     diff(q2,th,t)=Q20_3+diff(omega,th,r),#
     diff(q3,r,t)=Q30_2+diff(omega,th,r),#
     diff(q3,r,r)=-Q23_2+diff(q2,th,r),
     diff(q3,t,t)=Q30_0+diff(omega,th,t),
     diff(q2,t,t)=Q20_0+diff(omega,r,t)};

rS:={nu0=ln((1-2*M/r)^(1/2)),
     mu20=ln((1-2*M/r)^(-1/2)),
     mu30=ln(r),
     lambda0=ln(r*sin(th))};

Ricci1[1,4]:=simplify(subs(r2,Ricci1[1,4])):
Ricci1[1,4]:=simplify(subs(r1,Ricci1[1,4])):
Ricci1[1,4]:=simplify(subs(rS,Ricci1[1,4]));
Ricci1[1,2]:=simplify(subs(r2,Ricci1[1,2])):
Ricci1[1,2]:=simplify(subs(r1,Ricci1[1,2])):
Ricci1[1,2]:=simplify(subs(rS,Ricci1[1,2]));
Ricci1[1,3]:=simplify(subs(r2,Ricci1[1,3])):
Ricci1[1,3]:=simplify(subs(r1,Ricci1[1,3])):
Ricci1[1,3]:=simplify(subs(rS,Ricci1[1,3]));
Ricci1[3,1]:=simplify(subs(r2,Ricci1[3,1])):
Ricci1[3,1]:=simplify(subs(r1,Ricci1[3,1])):
Ricci1[3,1]:=simplify(subs(rS,Ricci1[3,1]));
Ricci1[2,3]:=simplify(subs(rS,Ricci1[2,3]));
Ricci1[2,2]:=simplify(subs(rS,Ricci1[2,2]));
Ricci1[3,3]:=simplify(subs(rS,Ricci1[3,3]));
Ricci1[4,4]:=simplify(subs(rS,Ricci1[4,4]));
Ricci1[1,1]:=simplify(subs(rS,Ricci1[1,1]));

ChR14:=-1/2*exp(-2*lambda0-mu20-mu30)* (
```

```
         Q20_2*exp(3*lambda0-nu0-mu20+mu30)
        +Q20*diff(exp(3*lambda0-nu0-mu20+mu30),r)
        +Q30_3*exp(3*lambda0-nu0+mu20-mu30)
        +Q30*diff(exp(3*lambda0-nu0+mu20-mu30),th)
             ):
ChR14:=simplify(subs(rS,ChR14));
checkR14:=simplify(ChR14/Ricci1[1,4]);

ChR12:=1/2*exp(-2*lambda0-nu0-mu30)* (
         Q23_3*exp(3*lambda0+nu0-mu20-mu30)
        +Q23*diff(exp(3*lambda0+nu0-mu20-mu30),th)
        -Q20_0*exp(3*lambda0-nu0-mu20+mu30)
        -Q30*diff(exp(3*lambda0-nu0-mu20+mu30),t)
             ):
ChR12:=simplify(subs(rS,ChR12));
checkR12:=simplify(ChR12/Ricci1[1,2]);

ChR13:=1/2*exp(-2*lambda0-nu0-mu20)* (
         -Q23_2*exp(3*lambda0+nu0-mu20-mu30)
        -Q23*diff(exp(3*lambda0+nu0-mu20-mu30),r)
        -Q30_0*exp(3*lambda0-nu0-mu30+mu20)
        -Q20*diff(exp(3*lambda0-nu0-mu30+mu20),t)
             ):
ChR13:=simplify(subs(rS,ChR13));
checkR13:=simplify(ChR13/Ricci1[1,3]);

ChR24:=exp(-mu2-nu)*(
        diff(lambda+mu3,t,r) + diff(lambda,r)*diff(lambda-mu2,t)
        +diff(mu3,r)*diff(mu3-mu2,t) - diff(lambda+mu3,t)*diff(nu,r)
         ):
```

```
ChR24_1:=factor(simplify(subs(eps=0,diff(ChR24,eps))));
checkR24:=simplify(ChR24_1/Ricci1[2,4]);

ChR23:=exp(-mu2-mu3)*(
         diff(lambda+nu,r,th) - diff(lambda+nu,r)*diff(mu2,th)
        -diff(lambda+nu,th)*diff(mu3,r) + diff(lambda,r)*diff(lambda,th)
        +diff(nu,r)*diff(nu,th)
         ):
ChR23:=simplify(subs(rS,ChR23));
ChR23_1:=factor(simplify(subs(eps=0,diff(ChR23,eps))));
checkR23:=factor(simplify(ChR23_1/Ricci1[2,3]));

ChR11:=exp(-2*mu2)*(diff(lambda,r,r)+diff(lambda,r)*diff(lambda+nu+mu3-mu2,r))
         +exp(-2*mu3)*(diff(lambda,th,th)+diff(lambda,th)*diff(lambda+nu+mu2-mu3,th))
         -exp(-2*nu)*(diff(lambda,t,t)+diff(lambda,t)*diff(lambda-nu+mu2+mu3,t)):
ChR11:=simplify(subs(rS,ChR11));
ChR11_1:=factor(simplify(subs(eps=0,diff(ChR11,eps))));
checkR11:=factor(simplify(ChR11_1/Ricci1[1,1]));

ChR22:=exp(-2*mu2)*(diff(lambda+nu+mu3,r,r)+diff(lambda,r)*diff(lambda-mu2,r)
         +diff(mu3,r)*diff(mu3-mu2,r)+diff(nu,r)*diff(nu-mu2,r))
        +exp(-2*mu3)*(diff(mu2,th,th)+diff(mu2,th)*diff(lambda+nu+mu2-mu3,th))
        -exp(-2*nu)*(diff(mu2,t,t)+diff(mu2,t)*diff(lambda-nu+mu2+mu3,t)):
ChR22:=simplify(subs(rS,ChR22));
ChR22_1:=factor(simplify(subs(eps=0,diff(ChR22,eps))));
checkR22:=factor(simplify(ChR22_1/Ricci1[2,2]));

ChR33:=exp(-2*mu3)*(diff(lambda+nu+mu2,th,th)+diff(lambda,th)*diff(lambda-mu3,th)
         +diff(mu2,th)*diff(mu2-mu3,th)+diff(nu,th)*diff(nu-mu3,th))
        +exp(-2*mu2)*(diff(mu3,r,r)+diff(mu3,r)*diff(lambda+nu+mu3-mu2,r))
```

```
       -exp(-2*nu)*(diff(mu3,t,t)+diff(mu3,t)*diff(lambda-nu+mu3+mu2,t)):
ChR33:=simplify(subs(rS,ChR33));
ChR33_1:=factor(simplify(subs(eps=0,diff(ChR33,eps))));
checkR33:=factor(simplify(ChR33_1/Ricci1[3,3]));

ChR44:=exp(-2*nu)*(diff(lambda+mu2+mu3,t,t)+diff(lambda,t)*diff(lambda-nu,t)
        +diff(mu2,t)*diff(mu2-nu,t)+diff(mu3,t)*diff(mu3-nu,t))
        -exp(-2*mu2)*(diff(nu,r,r)+diff(nu,r)*diff(lambda+nu-mu2+mu3,r))
        -exp(-2*mu3)*(diff(nu,th,th)+diff(nu,th)*diff(lambda+nu+mu2-mu3,th)):
ChR44:=simplify(subs(rS,ChR44));
ChR44_1:=factor(simplify(subs(eps=0,diff(ChR44,eps))));
checkR44:=factor(simplify(ChR44_1/Ricci1[4,4]));

ChR31:=1/2*exp(-2*lambda0-nu0-mu20)* (
        -Q23_2*exp(3*lambda0+nu0-mu20-mu30)
       -Q23*diff(exp(3*lambda0+nu0-mu20-mu30),r)
       -Q30_0*exp(3*lambda0-nu0-mu30+mu20)
       -Q20*diff(exp(3*lambda0-nu0-mu30+mu20),t)
            ):
ChR31:=simplify(subs(rS,ChR31));
checkR31:=simplify(ChR13/Ricci1[3,1]);
```

A.2 Program results

```
> read "c.map";
```

```
Warning, the protected names norm and trace have been redefined and
unprotected
```

$$gd := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$gu := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$x := [ph,\, r,\, th,\, t]$$

$$\mathit{gammad} := \mathrm{array}(1..4,\, 1..4,\, 1..4,\, [])$$

$$\mathit{gammau} := \mathrm{array}(1..4,\, 1..4,\, 1..4,\, [])$$

$$\mathit{gammaut} := \mathrm{array}(1..4,\, 1..4,\, 1..4,\, [])$$

$$\mathit{gammau0} := \mathrm{array}(1..4,\, 1..4,\, 1..4,\, [])$$

$$\mathit{gammau1} := \mathrm{array}(1..4,\, 1..4,\, 1..4,\, [])$$

$$\mathit{Riemann} := \mathrm{array}(1..4,\, 1..4,\, 1..4,\, 1..4,\, [])$$

$$\mathit{Ricci} := \mathrm{array}(1..4,\, 1..4,\, [])$$

$$\mathit{Ricci0} := \mathrm{array}(1..4,\, 1..4,\, [])$$

$$\mathit{Ricci1} := \mathrm{array}(1..4,\, 1..4,\, [])$$

$$\mathit{Ricci1}_{1,\,4} := -\frac{1}{2}(\sin(th)\; \mathit{Q20\_2}\; r^2 + 4\, r \sin(th)\; \mathit{Q20} - 2\, r \sin(th)\; \mathit{Q20\_2}\; M - 8 \sin(th)\; \mathit{Q20}\; M + \sin(th)\; \mathit{Q30\_3} + 3 \cos(th)\; \mathit{Q30}) \sin(th)$$

$$\mathit{Ricci1}_{1,\,2} := \frac{1}{2}(-\sin(th)\; \mathit{Q20\_0}\; r^3 + r \sin(th)\; \mathit{Q23\_3} + 3\, r \cos(th)\; \mathit{Q23} - 6 \cos(th)\; \mathit{Q23}\; M - 2 \sin(th)\; \mathit{Q23\_3}\; M) \sin(th)/(r - 2\, M)$$

$$\mathit{Ricci1}_{1,\,3} := -\frac{1}{2}(r^3\; \mathit{Q30\_0} + r^3\; \mathit{Q23\_2} + 2\, r^2\; \mathit{Q23} - 4\, r^2\; \mathit{Q23\_2}\; M - 6\, r\, M\; \mathit{Q23} + 4\, r\; \mathit{Q23\_2}\; M^2 + 4\, M^2\; \mathit{Q23}) \sin(th)^2/(r - 2\, M)$$

$$\mathit{Ricci1}_{3,\,1} := -\frac{1}{2}(r^3\; \mathit{Q30\_0} + r^3\; \mathit{Q23\_2} + 2\, r^2\; \mathit{Q23} - 4\, r^2\; \mathit{Q23\_2}\; M - 6\, r\, M\; \mathit{Q23} + 4\, r\; \mathit{Q23\_2}\; M^2 + 4\, M^2\; \mathit{Q23}) \sin(th)^2/(r - 2\, M)$$

$$\mathit{Ricci1}_{2,\,3} := (\cos(th)\, (\tfrac{\partial}{\partial r}\, \mu 31)\, r^2 - 2 \cos(th)\, (\tfrac{\partial}{\partial r}\, \mu 31)\, r\, M + (\tfrac{\partial}{\partial th}\, \mu 21)\, r \sin(th) - (\tfrac{\partial}{\partial th}\, \mu 21)\, M \sin(th) + (\tfrac{\partial}{\partial th}\, \nu 1)\, r \sin(th) - 3\, (\tfrac{\partial}{\partial th}\, \nu 1)\, M \sin(th) - (\tfrac{\partial}{\partial r}\, \lambda 1) \cos(th)\, r^2 + 2\, (\tfrac{\partial}{\partial r}\, \lambda 1) \cos(th)\, r\, M - (\tfrac{\partial^2}{\partial th\, \partial r}\, \lambda 1)\, r^2 \sin(th) + 2\, (\tfrac{\partial^2}{\partial th\, \partial r}\, \lambda 1)\, r\, M \sin(th) - (\tfrac{\partial^2}{\partial th\, \partial r}\, \nu 1)\, r^2 \sin(th) + 2\, (\tfrac{\partial^2}{\partial th\, \partial r}\, \nu 1)\, r\, M \sin(th))/(r \sin(th)\, (r - 2\, M))$$

$$Ricci1_{2,2} := -(2\,r^2\,(\frac{\partial}{\partial r}\,\lambda 1)\sin(th) - 2\,r^2\,(\frac{\partial}{\partial r}\,\mu 21)\sin(th) - 2\,(\frac{\partial}{\partial th}\,\mu 21)\cos(th)\,M$$
$$- 6\,M^2\,(\frac{\partial}{\partial r}\,\mu 21)\sin(th) + 6\,(\frac{\partial}{\partial r}\,\mu 31)\,M^2\sin(th) + 2\,r^2\,(\frac{\partial}{\partial r}\,\mu 31)\sin(th)$$
$$+ 6\,(\frac{\partial}{\partial r}\,\lambda 1)\,M^2\sin(th) - 6\,(\frac{\partial}{\partial r}\,\nu 1)\,M^2\sin(th) + r\,(\frac{\partial}{\partial th}\,\mu 21)\cos(th)$$
$$+ 3\,(\frac{\partial}{\partial r}\,\nu 1)\,M\,r\sin(th) + 7\,M\,r\,(\frac{\partial}{\partial r}\,\mu 21)\sin(th) - 7\,(\frac{\partial}{\partial r}\,\lambda 1)\,r\,M\sin(th)$$
$$- 7\,(\frac{\partial}{\partial r}\,\mu 31)\,r\,M\sin(th) - 4\,(\frac{\partial^2}{\partial r^2}\,\lambda 1)\,r^2\,M\sin(th) + (\frac{\partial^2}{\partial r^2}\,\mu 31)\,r^3\sin(th)$$
$$+ (\frac{\partial^2}{\partial r^2}\,\nu 1)\,r^3\sin(th) - r^3\,(\frac{\partial^2}{\partial t^2}\,\mu 21)\sin(th) + r\,(\frac{\partial^2}{\partial th^2}\,\mu 21)\sin(th)$$
$$- 2\,(\frac{\partial^2}{\partial th^2}\,\mu 21)\,M\sin(th) + (\frac{\partial^2}{\partial r^2}\,\lambda 1)\,r^3\sin(th) + 4\,(\frac{\partial^2}{\partial r^2}\,\nu 1)\,r\,M^2\sin(th)$$
$$+ 4\,(\frac{\partial^2}{\partial r^2}\,\mu 31)\,r\,M^2\sin(th) + 4\,(\frac{\partial^2}{\partial r^2}\,\lambda 1)\,r\,M^2\sin(th) - 4\,(\frac{\partial^2}{\partial r^2}\,\mu 31)\,r^2\,M\sin(th)$$
$$- 4\,(\frac{\partial^2}{\partial r^2}\,\nu 1)\,r^2\,M\sin(th))\Big/(\sin(th)\,r\,(r - 2\,M)^2)$$

$$Ricci1_{3,3} := (-r^2\,(\frac{\partial}{\partial r}\,\nu 1)\sin(th) - 2\,r\,\mu 31\sin(th) + 2\,r\,\mu 21\sin(th) + 4\cos(th)\,(\frac{\partial}{\partial th}\,\lambda 1)\,M$$
$$- 2\cos(th)\,(\frac{\partial}{\partial th}\,\lambda 1)\,r + (\frac{\partial}{\partial th}\,\mu 31)\cos(th)\,r + 4\,\mu 31\sin(th)\,M$$
$$- 2\,(\frac{\partial}{\partial th}\,\mu 31)\cos(th)\,M - 4\,\mu 21\sin(th)\,M + r^3\,(\frac{\partial^2}{\partial t^2}\,\mu 31)\sin(th)$$
$$- (\frac{\partial^2}{\partial th^2}\,\lambda 1)\,r\sin(th) - (\frac{\partial^2}{\partial th^2}\,\nu 1)\,r\sin(th) + 2\,(\frac{\partial^2}{\partial th^2}\,\lambda 1)\,M\sin(th)$$
$$+ 2\,(\frac{\partial^2}{\partial th^2}\,\nu 1)\,M\sin(th) - r^2\,(\frac{\partial}{\partial r}\,\lambda 1)\sin(th) + r^2\,(\frac{\partial}{\partial r}\,\mu 21)\sin(th)$$
$$+ 4\,M^2\,(\frac{\partial}{\partial r}\,\mu 21)\sin(th) - 8\,(\frac{\partial}{\partial r}\,\mu 31)\,M^2\sin(th) - 3\,r^2\,(\frac{\partial}{\partial r}\,\mu 31)\sin(th)$$
$$- 4\,(\frac{\partial}{\partial r}\,\lambda 1)\,M^2\sin(th) - 4\,(\frac{\partial}{\partial r}\,\nu 1)\,M^2\sin(th) + 4\,(\frac{\partial}{\partial r}\,\nu 1)\,M\,r\sin(th)$$
$$- 4\,M\,r\,(\frac{\partial}{\partial r}\,\mu 21)\sin(th) + 4\,(\frac{\partial}{\partial r}\,\lambda 1)\,r\,M\sin(th) + 10\,(\frac{\partial}{\partial r}\,\mu 31)\,r\,M\sin(th)$$
$$- (\frac{\partial^2}{\partial r^2}\,\mu 31)\,r^3\sin(th) - r\,(\frac{\partial^2}{\partial th^2}\,\mu 21)\sin(th) + 2\,(\frac{\partial^2}{\partial th^2}\,\mu 21)\,M\sin(th)$$
$$- 4\,(\frac{\partial^2}{\partial r^2}\,\mu 31)\,r\,M^2\sin(th) + 4\,(\frac{\partial^2}{\partial r^2}\,\mu 31)\,r^2\,M\sin(th))/(\sin(th)\,(r - 2\,M))$$

$$Ricci1_{4,4} := -(-2\,r^2\,(\frac{\partial}{\partial r}\,\nu 1)\sin(th) - (\frac{\partial^2}{\partial th^2}\,\nu 1)\,r\sin(th) + 2\,(\frac{\partial^2}{\partial th^2}\,\nu 1)\,M\sin(th)$$
$$- (\frac{\partial}{\partial th}\,\nu 1)\cos(th)\,r + 2\,(\frac{\partial}{\partial th}\,\nu 1)\cos(th)\,M - (\frac{\partial}{\partial r}\,\lambda 1)\,r\,M\sin(th)$$
$$+ 2\,(\frac{\partial}{\partial r}\,\lambda 1)\,M^2\sin(th) + M\,r\,(\frac{\partial}{\partial r}\,\mu 21)\sin(th) - 2\,M^2\,(\frac{\partial}{\partial r}\,\mu 21)\sin(th)$$
$$- 2\,(\frac{\partial}{\partial r}\,\nu 1)\,M^2\sin(th) + 5\,(\frac{\partial}{\partial r}\,\nu 1)\,M\,r\sin(th) - (\frac{\partial}{\partial r}\,\mu 31)\,r\,M\sin(th)$$
$$+ 2\,(\frac{\partial}{\partial r}\,\mu 31)\,M^2\sin(th) + 4\,(\frac{\partial^2}{\partial r^2}\,\nu 1)\,r^2\,M\sin(th) - 4\,(\frac{\partial^2}{\partial r^2}\,\nu 1)\,r\,M^2\sin(th)$$
$$+ r^3\,(\frac{\partial^2}{\partial t^2}\,\mu 31)\sin(th) + r^3\,(\frac{\partial^2}{\partial t^2}\,\mu 21)\sin(th) + (\frac{\partial^2}{\partial t^2}\,\lambda 1)\,r^3\sin(th) - (\frac{\partial^2}{\partial r^2}\,\nu 1)\,r^3\sin(th)$$
$$)/(r^3\sin(th))$$

$$
\begin{aligned}
\mathit{Ricci1}_{1,1} := {} & (2\,(\tfrac{\partial^2}{\partial th^2}\,\lambda 1)\,M\sin(th) - (\tfrac{\partial}{\partial th}\,\nu 1)\cos(th)\,r + 2\,(\tfrac{\partial}{\partial th}\,\nu 1)\cos(th)\,M \\
& - r^2\,(\tfrac{\partial}{\partial r}\,\nu 1)\sin(th) - 2\,r\,\mu 31\sin(th) + 2\,r\,\mu 21\sin(th) + 4\cos(th)\,(\tfrac{\partial}{\partial th}\,\lambda 1)\,M \\
& - 2\cos(th)\,(\tfrac{\partial}{\partial th}\,\lambda 1)\,r + (\tfrac{\partial}{\partial th}\,\mu 31)\cos(th)\,r + 4\,\mu 31\sin(th)\,M \\
& - 2\,(\tfrac{\partial}{\partial th}\,\mu 31)\cos(th)\,M - 4\,\mu 21\sin(th)\,M - (\tfrac{\partial^2}{\partial th^2}\,\lambda 1)\,r\sin(th) \\
& - (\tfrac{\partial^2}{\partial r^2}\,\lambda 1)\,r^3\sin(th) + 4\,(\tfrac{\partial^2}{\partial r^2}\,\lambda 1)\,r^2\,M\sin(th) - 4\,(\tfrac{\partial^2}{\partial r^2}\,\lambda 1)\,r\,M^2\sin(th) \\
& - 3\,r^2\,(\tfrac{\partial}{\partial r}\,\lambda 1)\sin(th) + r^2\,(\tfrac{\partial}{\partial r}\,\mu 21)\sin(th) + 2\,(\tfrac{\partial}{\partial th}\,\mu 21)\cos(th)\,M \\
& + 4\,M^2\,(\tfrac{\partial}{\partial r}\,\mu 21)\sin(th) - 4\,(\tfrac{\partial}{\partial r}\,\mu 31)\,M^2\sin(th) - r^2\,(\tfrac{\partial}{\partial r}\,\mu 31)\sin(th) \\
& - 8\,(\tfrac{\partial}{\partial r}\,\lambda 1)\,M^2\sin(th) - 4\,(\tfrac{\partial}{\partial r}\,\nu 1)\,M^2\sin(th) - r\,(\tfrac{\partial}{\partial th}\,\mu 21)\cos(th) \\
& + 4\,(\tfrac{\partial}{\partial r}\,\nu 1)\,M\,r\sin(th) - 4\,M\,r\,(\tfrac{\partial}{\partial r}\,\mu 21)\sin(th) + 10\,(\tfrac{\partial}{\partial r}\,\lambda 1)\,r\,M\sin(th) \\
& + 4\,(\tfrac{\partial}{\partial r}\,\mu 31)\,r\,M\sin(th) + (\tfrac{\partial^2}{\partial t^2}\,\lambda 1)\,r^3\sin(th))\sin(th)/(r - 2\,M)
\end{aligned}
$$

$$\mathit{checkR14} := \frac{\sqrt{\frac{r-2\,M}{r}}}{(r-2\,M)\sin(th)}$$

$$\mathit{checkR12} := \frac{r-2\,M}{r^2\,\sqrt{\frac{r-2\,M}{r}}\,\sin(th)}$$

$$\mathit{checkR13} := -\frac{\sin(th)}{r^2\,(-1+\cos(th)^2)}$$

$$\mathit{checkR24} := -e^{(-\mu 20-\nu 0)}$$

$$\mathit{checkR23} := -\frac{\sqrt{\frac{r-2\,M}{r}}}{r}$$

$$\mathit{checkR11} := \frac{1}{r^2\,(\cos(th)-1)\,(\cos(th)+1)}$$

$$\mathit{checkR22} := -\frac{r-2\,M}{r}$$

$$\mathit{checkR33} := -\frac{1}{r^2}$$

$$\mathit{checkR44} := -\frac{r}{r-2\,M}$$

$$\mathit{checkR31} := -\frac{\sin(th)}{r^2\,(-1+\cos(th)^2)}$$

A.3 Program: c2.map

This program calculates the non-linear Ricci tensor R44 of non-axisymmetric space-times of chapter 3 sec. 3.2. It then checks R44 against the original non-linear version calculated by Chandrasekhar in 1972.

```
with(linalg):

#Matrix and array declarations gd:=matrix(4,4);        #$g_{ab}$
gu:=matrix(4,4);        #$g^{ab}$ x:=array([ph,r,th,t]);  #$x^a$
gammad:=array(1..4,1..4,1..4);   #$\Gamma_{abc}$
gammau:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
Riemann:=array(1..4,1..4,1..4,1..4);#$R^a_{bcd}$ Ricci:=matrix(4,4);
#$R_{ab}$

alias(lambda=lambda(r,th,t)); alias(omega=omega(r,th,t));
alias(mu2=m2(r,th,t)); alias(mu3=m3(r,th,t)); alias(nu=nu(r,th,t));
alias(q2=q2(r,th,t)); alias(q3=q3(r,th,t));

gd[2,2]:=-(exp(2*lambda)*q2^2+exp(2*mu2)):
gd[2,3]:=-exp(2*lambda)*q2*q3:gd[3,2]:=gd[2,3]:
gd[1,2]:=exp(2*lambda)*q2:gd[2,1]:=gd[1,2]:
gd[2,4]:=-exp(2*lambda)*q2*omega:gd[4,2]:=gd[2,4]:
gd[3,3]:=-(exp(2*lambda)*q3^2+exp(2*mu3)):
gd[1,3]:=exp(2*lambda)*q3:gd[3,1]:=gd[1,3]:
gd[3,4]:=-exp(2*lambda)*q3*omega:gd[4,3]:=gd[3,4]:
gd[1,1]:=-exp(2*lambda):
gd[1,4]:=exp(2*lambda)*omega:gd[4,1]:=gd[1,4]:
gd[4,4]:=(exp(2*nu)-exp(2*lambda)*omega^2):
```

```
print(gd);

gu:=inverse(gd);

for a1 from 1 to 4 do for b1 from 1 to 4 do for c1 from 1 to 4 do
  gammad[a1,b1,c1]:=(diff(gd[a1,b1],x[c1]) + diff(gd[c1,a1],x[b1])
                     - diff(gd[b1,c1],x[a1]))/2;
od; od; od; for a1 from 1 to 4 do for b1 from 1 to 4 do for c1 from
1 to 4 do
  gammau[a1,b1,c1]:=simplify(sum(gu[a1,d1]*gammad[d1,b1,c1],d1=1..4));
od; od; od;

for a1 from 1 to 4 do for b1 from 1 to 4 do
   for c1 from 1 to 4 do for d1 from 1 to 4 do
Riemann[a1,b1,c1,d1]:= diff(gammau[a1,b1,d1],x[c1]) -
diff(gammau[a1,b1,c1],x[d1])+
sum(gammau[a1,e1,c1]*gammau[e1,b1,d1]-gammau[a1,e1,d1]*gammau[e1,b1,c1],e1=1..4);
od; od; od; od;

r1:={diff(q2,t)=Q20+diff(omega,r), diff(q3,t)=Q30+diff(omega,th),
diff(q2,th)=Q23+diff(q3,r)};
r2:={diff(q2,r,t)=Q20_2+diff(omega,r,r),
     diff(q3,t,th)=Q30_3+diff(omega,th,th),
     diff(q2,th,th)=Q23_3+diff(q3,th,r),
     diff(q2,th,t)=Q20_3+diff(omega,th,r),
     diff(q3,r,t)=Q30_2+diff(omega,th,r),
     diff(q3,r,r)=-Q23_2+diff(q2,th,r),
     diff(q3,t,t)=Q30_0+diff(omega,th,t),
     diff(q2,t,t)=Q20_0+diff(omega,r,t)};
```

```
Ricci[4,4]:=simplify(sum(Riemann[e1,4,e1,4],e1=1..4));
Ricci[4,4]:=simplify(subs(r2,Ricci[4,4])):
Ricci[4,4]:=simplify(subs(r1,Ricci[4,4])):
ChR44:=exp(-2*nu)*(diff(lambda+mu2+mu3,t,t)+diff(lambda,t)*diff(lambda-nu,t)
        +diff(mu2,t)*diff(mu2-nu,t)+diff(mu3,t)*diff(mu3-nu,t))
        -exp(-2*mu2)*(diff(nu,r,r)+diff(nu,r)*diff(lambda+nu-mu2+mu3,r))
        -exp(-2*mu3)*(diff(nu,th,th)+diff(nu,th)*diff(lambda+nu+mu2-mu3,th))
        +1/2*exp(2*lambda-2*nu)*(exp(-2*mu2)*Q20^2+exp(-2*mu3)*Q30^2):
checkR44:=-exp(-2*nu):
checkRt44:=factor(simplify(ChR44-checkR44*Ricci[4,4]));

RiciD44:=1/2*omega*(exp(-4*nu+2*lambda)*(2*omega*(diff(lambda,t,t)
         +diff(lambda,t)*diff(lambda+mu3+mu2-nu,t)))
         +exp(-2*nu-2*mu2+2*lambda)*(-2*omega*(diff(lambda,r,r)
         +diff(lambda,r)*diff(lambda+mu3-mu2+nu,r))
         +2*Q20*diff(mu3-mu2-nu+3*lambda,r)+2*Q20_2)
         +exp(-2*nu-2*mu3+2*lambda)*(-2*omega*(diff(lambda,th,th)
         +diff(lambda,th)*diff(lambda+mu2+nu-mu3,th))
         +2*Q30*diff(-mu3+mu2+3*lambda-nu,th)+2*Q30_3)
         -omega*Q20^2*exp(-4*nu+4*lambda-2*mu2)
         -omega*Q30^2*exp(-4*nu-2*mu3+4*lambda)
         +omega*Q23^2*exp(-2*nu-2*mu2+4*lambda-2*mu3));
CheckDiference:=factor(simplify(checkRt44-RiciD44));
```

A.4 Program results

```
> read "c2.map";
```

```
Warning, the protected names norm and trace have been redefined and
unprotected
```

$$gd := \text{array}(1..4,\ 1..4,\ [])$$

$$gu := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$x := [ph,\ r,\ th,\ t]$$

$$gammad := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ [])$$

$$gammau := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ [])$$

$$Riemann := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ 1..4,\ [])$$

$$Ricci := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$\lambda$$

$$\lambda,\ \omega$$

$$\lambda,\ \omega,\ \mu 2$$

$$\lambda,\ \omega,\ \mu 2,\ \mu 3$$

$$\lambda,\ \omega,\ \mu 2,\ \mu 3,\ \nu$$

$$\lambda,\ \omega,\ \mu 2,\ \mu 3,\ \nu,\ q2$$

$$\lambda,\ \omega,\ \mu 2,\ \mu 3,\ \nu,\ q2,\ q3$$

$$\begin{bmatrix} -e^{(2\,\lambda)} & e^{(2\,\lambda)}\,q2 & e^{(2\,\lambda)}\,q3 & e^{(2\,\lambda)}\,\omega \\ e^{(2\,\lambda)}\,q2 & -e^{(2\,\lambda)}\,q2^2 - e^{(2\,\mu 2)} & -e^{(2\,\lambda)}\,q2\,q3 & -e^{(2\,\lambda)}\,q2\,\omega \\ e^{(2\,\lambda)}\,q3 & -e^{(2\,\lambda)}\,q2\,q3 & -e^{(2\,\lambda)}\,q3^2 - e^{(2\,\mu 3)} & -e^{(2\,\lambda)}\,q3\,\omega \\ e^{(2\,\lambda)}\,\omega & -e^{(2\,\lambda)}\,q2\,\omega & -e^{(2\,\lambda)}\,q3\,\omega & e^{(2\,\nu)} - e^{(2\,\lambda)}\,\omega^2 \end{bmatrix}$$

$gu :=$

$$\left[-\frac{e^{(2\,\lambda)}\,q2^2\,e^{(2\,\mu 3)}\,e^{(2\,\nu)} + e^{(2\,\mu 2)}\,e^{(2\,\lambda)}\,q3^2\,e^{(2\,\nu)} + e^{(2\,\mu 2)}\,e^{(2\,\mu 3)}\,e^{(2\,\nu)} - e^{(2\,\mu 2)}\,e^{(2\,\mu 3)}\,e^{(2\,\lambda)}\,\omega^2}{e^{(2\,\lambda)}\,e^{(2\,\mu 2)}\,e^{(2\,\mu 3)}\,e^{(2\,\nu)}}\right.$$

$$\left., -\frac{q2}{e^{(2\,\mu 2)}}, -\frac{q3}{e^{(2\,\mu 3)}}, \frac{\omega}{e^{(2\,\nu)}}\right]$$

$$\left[-\frac{q2}{e^{(2\,\mu 2)}}, -\frac{1}{e^{(2\,\mu 2)}}, 0, 0\right]$$

$$\left[-\frac{q3}{e^{(2\,\mu 3)}}, 0, -\frac{1}{e^{(2\,\mu 3)}}, 0\right]$$

$$\left[\frac{\omega}{e^{(2\,\nu)}}, 0, 0, \frac{1}{e^{(2\,\nu)}}\right]$$

$$
\begin{aligned}
Ricci_{4,4} :=\ & \%2\,(\tfrac{\partial^2}{\partial t^2}\,\lambda)\,\omega^2 - (\tfrac{\partial}{\partial r}\,\mu 2)\,\%5\,(\tfrac{\partial}{\partial r}\,\nu) - \frac{1}{2}\,\omega^2\,e^{(-2\,\nu+4\,\lambda-2\,\mu 2)}\,(\tfrac{\partial}{\partial r}\,\omega)^2 \\
& - \frac{1}{2}\,\omega^2\,e^{(-2\,\nu+4\,\lambda-2\,\mu 2)}\,(\tfrac{\partial}{\partial t}\,q2)^2 - e^{(-2\,\mu 2+4\,\lambda-2\,\mu 3)}\,(\tfrac{\partial}{\partial th}\,q2)\,\omega^2\,(\tfrac{\partial}{\partial r}\,q3) \\
& - \frac{1}{2}\,e^{(-2\,\mu 3+4\,\lambda-2\,\nu)}\,(\tfrac{\partial}{\partial t}\,q3)^2\,\omega^2 + \frac{1}{2}\,e^{(-2\,\mu 2+4\,\lambda-2\,\mu 3)}\,(\tfrac{\partial}{\partial r}\,q3)^2\,\omega^2 \\
& - \frac{1}{2}\,e^{(-2\,\mu 3+4\,\lambda-2\,\nu)}\,\omega^2\,(\tfrac{\partial}{\partial th}\,\omega)^2 + \%5\,(\tfrac{\partial^2}{\partial r^2}\,\nu) - (\tfrac{\partial^2}{\partial t^2}\,\mu 2) - (\tfrac{\partial^2}{\partial t^2}\,\mu 3) + \%4\,(\tfrac{\partial^2}{\partial th^2}\,\nu) \\
& - \%3\,(\tfrac{\partial^2}{\partial r^2}\,\lambda)\,\omega^2 - \%3\,\omega\,(\tfrac{\partial^2}{\partial r^2}\,\omega) + \%3\,(\tfrac{\partial^2}{\partial t\,\partial r}\,q2)\,\omega - \%1\,(\tfrac{\partial^2}{\partial th^2}\,\lambda)\,\omega^2 - (\tfrac{\partial^2}{\partial t^2}\,\lambda) \\
& - \frac{1}{2}\,\%3\,(\tfrac{\partial}{\partial r}\,\omega)^2 - 3\,\%3\,(\tfrac{\partial}{\partial r}\,\lambda)\,\omega\,(\tfrac{\partial}{\partial r}\,\omega) + 3\,(\tfrac{\partial}{\partial t}\,q3)\,\%1\,(\tfrac{\partial}{\partial th}\,\lambda)\,\omega - \%1\,\omega\,(\tfrac{\partial^2}{\partial th^2}\,\omega) \\
& + \%1\,(\tfrac{\partial^2}{\partial th\,\partial t}\,q3)\,\omega + 3\,(\tfrac{\partial}{\partial t}\,q2)\,\%3\,(\tfrac{\partial}{\partial r}\,\lambda)\,\omega + (\tfrac{\partial}{\partial t}\,q3)\,\%1\,(\tfrac{\partial}{\partial th}\,\omega) \\
& - 3\,\%1\,(\tfrac{\partial}{\partial th}\,\lambda)\,\omega\,(\tfrac{\partial}{\partial th}\,\omega) - \frac{1}{2}\,(\tfrac{\partial}{\partial t}\,q3)^2\,\%1 + (\tfrac{\partial}{\partial t}\,q2)\,\%3\,(\tfrac{\partial}{\partial r}\,\omega) - \frac{1}{2}\,(\tfrac{\partial}{\partial t}\,q2)^2\,\%3 \\
& + \omega^2\,e^{(-2\,\nu+4\,\lambda-2\,\mu 2)}\,(\tfrac{\partial}{\partial r}\,\omega)\,(\tfrac{\partial}{\partial t}\,q2) + \frac{1}{2}\,e^{(-2\,\mu 2+4\,\lambda-2\,\mu 3)}\,(\tfrac{\partial}{\partial th}\,q2)^2\,\omega^2 \\
& + \omega\,\%3\,(\tfrac{\partial}{\partial r}\,\omega)\,(\tfrac{\partial}{\partial r}\,\nu) - \omega\,\%3\,(\tfrac{\partial}{\partial t}\,q2)\,(\tfrac{\partial}{\partial r}\,\nu) - \%1\,(\tfrac{\partial}{\partial th}\,\lambda)\,\omega^2\,(\tfrac{\partial}{\partial th}\,\nu) \\
& + (\tfrac{\partial}{\partial t}\,\mu 3)\,\%2\,(\tfrac{\partial}{\partial t}\,\lambda)\,\omega^2 - \%1\,(\tfrac{\partial}{\partial th}\,\lambda)^2\,\omega^2 - (\tfrac{\partial}{\partial th}\,\mu 3)\,\%4\,(\tfrac{\partial}{\partial th}\,\nu) \\
& + (\tfrac{\partial}{\partial r}\,\mu 3)\,\%5\,(\tfrac{\partial}{\partial r}\,\nu) - \%3\,(\tfrac{\partial}{\partial r}\,\lambda)^2\,\omega^2 + (\tfrac{\partial}{\partial r}\,\lambda)\,\%5\,(\tfrac{\partial}{\partial r}\,\nu) + (\tfrac{\partial}{\partial th}\,\lambda)\,\%4\,(\tfrac{\partial}{\partial th}\,\nu) \\
& + \%2\,(\tfrac{\partial}{\partial t}\,\lambda)^2\,\omega^2 + (\tfrac{\partial}{\partial th}\,\mu 2)\,\%4\,(\tfrac{\partial}{\partial th}\,\nu) - (\tfrac{\partial}{\partial th}\,\mu 3)\,\%1\,(\tfrac{\partial}{\partial t}\,q3)\,\omega \\
& + (\tfrac{\partial}{\partial th}\,\mu 3)\,\%1\,(\tfrac{\partial}{\partial th}\,\lambda)\,\omega^2 + (\tfrac{\partial}{\partial th}\,\mu 3)\,\%1\,\omega\,(\tfrac{\partial}{\partial th}\,\omega) + (\tfrac{\partial}{\partial r}\,\mu 3)\,\%3\,(\tfrac{\partial}{\partial t}\,q2)\,\omega \\
& - (\tfrac{\partial}{\partial r}\,\mu 3)\,\%3\,(\tfrac{\partial}{\partial r}\,\lambda)\,\omega^2 - (\tfrac{\partial}{\partial r}\,\mu 3)\,\%3\,\omega\,(\tfrac{\partial}{\partial r}\,\omega) - (\tfrac{\partial}{\partial t}\,\mu 3)^2 - (\tfrac{\partial}{\partial t}\,\lambda)^2 - (\tfrac{\partial}{\partial t}\,\mu 2)^2 \\
& + (\tfrac{\partial}{\partial t}\,\mu 3)\,(\tfrac{\partial}{\partial t}\,\nu) + \%5\,(\tfrac{\partial}{\partial r}\,\nu)^2 + \%4\,(\tfrac{\partial}{\partial th}\,\nu)^2 + (\tfrac{\partial}{\partial t}\,\mu 2)\,(\tfrac{\partial}{\partial t}\,\nu) - \%1\,(\tfrac{\partial}{\partial th}\,\nu)\,\omega\,(\tfrac{\partial}{\partial t}\,q3) \\
& - (\tfrac{\partial}{\partial r}\,\mu 2)\,\%3\,(\tfrac{\partial}{\partial t}\,q2)\,\omega + (\tfrac{\partial}{\partial th}\,\mu 2)\,\%1\,(\tfrac{\partial}{\partial t}\,q3)\,\omega - (\tfrac{\partial}{\partial th}\,\mu 2)\,\%1\,(\tfrac{\partial}{\partial th}\,\lambda)\,\omega^2 \\
& - (\tfrac{\partial}{\partial th}\,\mu 2)\,\%1\,\omega\,(\tfrac{\partial}{\partial th}\,\omega) + (\tfrac{\partial}{\partial t}\,\mu 2)\,\%2\,(\tfrac{\partial}{\partial t}\,\lambda)\,\omega^2 + \%3\,\omega\,(\tfrac{\partial}{\partial r}\,\omega)\,(\tfrac{\partial}{\partial r}\,\mu 2) \\
& + \%3\,(\tfrac{\partial}{\partial r}\,\lambda)\,\omega^2\,(\tfrac{\partial}{\partial r}\,\mu 2) - \omega^2\,(\tfrac{\partial}{\partial r}\,\nu)\,\%3\,(\tfrac{\partial}{\partial r}\,\lambda) - \omega^2\,(\tfrac{\partial}{\partial t}\,\lambda)\,\%2\,(\tfrac{\partial}{\partial t}\,\nu) + (\tfrac{\partial}{\partial t}\,\lambda)\,(\tfrac{\partial}{\partial t}\,\nu) \\
& + \%1\,(\tfrac{\partial}{\partial th}\,\nu)\,\omega\,(\tfrac{\partial}{\partial th}\,\omega) + e^{(-2\,\mu 3+4\,\lambda-2\,\nu)}\,(\tfrac{\partial}{\partial t}\,q3)\,\omega^2\,(\tfrac{\partial}{\partial th}\,\omega) - \frac{1}{2}\,\%1\,(\tfrac{\partial}{\partial th}\,\omega)^2
\end{aligned}
$$

$\%1 := e^{(-2\,\mu 3+2\,\lambda)}$

$\%2 := e^{(-2\,\nu+2\,\lambda)}$

$\%3 := e^{(-2\,\mu 2+2\,\lambda)}$

$\%4 := e^{(-2\,\mu 3+2\,\nu)}$

$\%5 := e^{(-2\,\mu 2+2\,\nu)}$

$$
\begin{aligned}
\mathit{checkRt44} := \frac{1}{2}\omega(&-2\,\omega\,(\tfrac{\partial}{\partial r}\,\lambda)\,(\tfrac{\partial}{\partial r}\,\mu3)\,\%3 + 2\,\mathit{Q20\_2}\,\%3 + 2\,\%2\,\mathit{Q30\_3}\\
&+ 2\,\omega\,(\tfrac{\partial}{\partial t}\,\lambda)\,(\tfrac{\partial}{\partial t}\,\mu3)\,\%1 + 2\,\omega\,(\tfrac{\partial}{\partial r}\,\mu2)\,(\tfrac{\partial}{\partial r}\,\lambda)\,\%3 + 2\,\omega\,(\tfrac{\partial}{\partial t}\,\lambda)\,(\tfrac{\partial}{\partial t}\,\mu2)\,\%1\\
&- 2\,\omega\,(\tfrac{\partial}{\partial r}\,\lambda)\,(\tfrac{\partial}{\partial r}\,\nu)\,\%3 - 2\,\omega\,(\tfrac{\partial}{\partial t}\,\nu)\,(\tfrac{\partial}{\partial t}\,\lambda)\,\%1 - 2\,\omega\,\%2\,(\tfrac{\partial}{\partial th}\,\mu2)\,(\tfrac{\partial}{\partial th}\,\lambda)\\
&- 2\,\omega\,\%2\,(\tfrac{\partial}{\partial th}\,\lambda)\,(\tfrac{\partial}{\partial th}\,\nu) + 2\,\omega\,\%2\,(\tfrac{\partial}{\partial th}\,\mu3)\,(\tfrac{\partial}{\partial th}\,\lambda)\\
&+ \omega\,\mathit{Q23}^2\,e^{(-2\,\nu-2\,\mu2+4\,\lambda-2\,\mu3)} - \omega\,\mathit{Q20}^2\,e^{(-4\,\nu+4\,\lambda-2\,\mu2)} - 2\,\omega\,\%2\,(\tfrac{\partial}{\partial th}\,\lambda)^2\\
&- \omega\,\mathit{Q30}^2\,e^{(-4\,\nu-2\,\mu3+4\,\lambda)} - 2\,\omega\,(\tfrac{\partial}{\partial r}\,\lambda)^2\,\%3 - 2\,\%2\,(\tfrac{\partial}{\partial th}\,\mu3)\,\mathit{Q30}\\
&+ 2\,\%2\,(\tfrac{\partial}{\partial th}\,\mu2)\,\mathit{Q30} + 6\,\%2\,(\tfrac{\partial}{\partial th}\,\lambda)\,\mathit{Q30} + 2\,\mathit{Q20}\,(\tfrac{\partial}{\partial r}\,\mu3)\,\%3\\
&- 2\,\mathit{Q20}\,(\tfrac{\partial}{\partial r}\,\mu2)\,\%3 - 2\,\mathit{Q20}\,(\tfrac{\partial}{\partial r}\,\nu)\,\%3 + 6\,\mathit{Q20}\,(\tfrac{\partial}{\partial r}\,\lambda)\,\%3 + 2\,\omega\,(\tfrac{\partial^2}{\partial t^2}\,\lambda)\,\%1\\
&- 2\,\omega\,\%2\,(\tfrac{\partial^2}{\partial th^2}\,\lambda) - 2\,\omega\,(\tfrac{\partial^2}{\partial r^2}\,\lambda)\,\%3 - 2\,\%2\,(\tfrac{\partial}{\partial th}\,\nu)\,\mathit{Q30} + 2\,\omega\,(\tfrac{\partial}{\partial t}\,\lambda)^2\,\%1)\\
\%1 &:= e^{(-4\,\nu+2\,\lambda)}\\
\%2 &:= e^{(-2\,\nu-2\,\mu3+2\,\lambda)}\\
\%3 &:= e^{(-2\,\nu-2\,\mu2+2\,\lambda)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{RiciD44} := \frac{1}{2}\omega(2\,e^{(-4\,\nu+2\,\lambda)}\,\omega\,((\tfrac{\partial^2}{\partial t^2}\,\lambda) + (\tfrac{\partial}{\partial t}\,\lambda)\,((\tfrac{\partial}{\partial t}\,\lambda) + (\tfrac{\partial}{\partial t}\,\mu3) + (\tfrac{\partial}{\partial t}\,\mu2) - (\tfrac{\partial}{\partial t}\,\nu)))+\\
&e^{(-2\,\nu-2\,\mu2+2\,\lambda)}(-2\,\omega\,((\tfrac{\partial^2}{\partial r^2}\,\lambda) + (\tfrac{\partial}{\partial r}\,\lambda)\,((\tfrac{\partial}{\partial r}\,\lambda) + (\tfrac{\partial}{\partial r}\,\nu) - (\tfrac{\partial}{\partial r}\,\mu2) + (\tfrac{\partial}{\partial r}\,\mu3)))\\
&+ 2\,\mathit{Q20}\,((\tfrac{\partial}{\partial r}\,\mu3) - (\tfrac{\partial}{\partial r}\,\mu2) - (\tfrac{\partial}{\partial r}\,\nu) + 3\,(\tfrac{\partial}{\partial r}\,\lambda)) + 2\,\mathit{Q20\_2}) + e^{(-2\,\nu-2\,\mu3+2\,\lambda)}(\\
&-2\,\omega\,((\tfrac{\partial^2}{\partial th^2}\,\lambda) + (\tfrac{\partial}{\partial th}\,\lambda)\,((\tfrac{\partial}{\partial th}\,\lambda) + (\tfrac{\partial}{\partial th}\,\nu) + (\tfrac{\partial}{\partial th}\,\mu2) - (\tfrac{\partial}{\partial th}\,\mu3)))\\
&+ 2\,\mathit{Q30}\,(-(\tfrac{\partial}{\partial th}\,\mu3) + (\tfrac{\partial}{\partial th}\,\mu2) + 3\,(\tfrac{\partial}{\partial th}\,\lambda) - (\tfrac{\partial}{\partial th}\,\nu)) + 2\,\mathit{Q30\_3})\\
&- \omega\,\mathit{Q20}^2\,e^{(-4\,\nu+4\,\lambda-2\,\mu2)} - \omega\,\mathit{Q30}^2\,e^{(-4\,\nu-2\,\mu3+4\,\lambda)}\\
&+ \omega\,\mathit{Q23}^2\,e^{(-2\,\nu-2\,\mu2+4\,\lambda-2\,\mu3)})
\end{aligned}
$$

$$
\mathit{CheckDiference} := 0
$$

Appendix B

Maple computer program 2

B.1 Program: t.map

This program calculates the metric connections, Riemann tensors, Ricci tensors and Ricci scalars of chapter 3 sec. 3.3.2.

```
with(linalg):

#Matrix and array declarations gd:=matrix(4,4);        #$g_{ab}$
gu:=matrix(4,4);        #$g^{ab}$ x:=array([r,th,ph,t]);  #$x^a$
gammad:=array(1..4,1..4,1..4);   #$\Gamma_{abc}$
gammau:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
gammaut:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
gammau0:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
gammau1:=array(1..4,1..4,1..4);   #$\Gamma^a_{bc}$
Riemann:=array(1..4,1..4,1..4,1..4);#$R^a_{bcd}$ Ricci:=matrix(4,4);
#$R_{ab}$ Ricci0:=matrix(4,4);      #$R_{ab}$ Ricci1:=matrix(4,4);
#$R_{ab}$

P2:=(3*cos(th)^2-1)/2; P2_t:=diff(P2,th); P2_tt:=diff(P2_t,th);
```

```
#V(r):=T(r)+L(r); gd[1,1]:=1/(1-2*m/r)*(1+eps*exp(I*s*t)*2*P2*L(r)):
gd[1,2]:=0:gd[2,1]:=gd[1,2]: gd[1,3]:=0:gd[3,1]:=gd[1,3]:
gd[1,4]:=0:gd[4,1]:=gd[1,4]:
gd[2,2]:=r^2*(1+eps*exp(I*s*t)*2*(P2*T(r)+P2_tt*V(r))):
gd[2,3]:=0:gd[3,2]:=gd[2,3]: gd[2,4]:=0:gd[4,2]:=gd[2,4]:
gd[3,3]:=r^2*sin(th)^2*(1+eps*exp(I*s*t)*2*(P2*T(r)+P2_t*cot(th)*V(r))):
gd[3,4]:=0:gd[4,3]:=gd[3,4]:
gd[4,4]:=-(1-2*m/r)*(1+eps*exp(I*s*t)*(3*cos(th)^2-1)*N(r)):
print(gd);

gu:=inverse(gd);

for a1 from 1 to 4 do for b1 from 1 to 4 do for c1 from 1 to 4 do
  gammad[a1,b1,c1]:=(diff(gd[a1,b1],x[c1]) + diff(gd[c1,a1],x[b1])
                     - diff(gd[b1,c1],x[a1]))/2;
od; od; od;

for a1 from 1 to 4 do for b1 from 1 to 4 do for c1 from 1 to 4 do
  gammaut[a1,b1,c1]:=simplify(sum(gu[a1,d1]*gammad[d1,b1,c1],d1=1..4));
gammau0[a1,b1,c1]:=simplify(subs(eps=0,gammaut[a1,b1,c1]));
gammau1[a1,b1,c1]:=simplify(subs(eps=0,diff(gammaut[a1,b1,c1],eps)));
gammau[a1,b1,c1]:=gammau0[a1,b1,c1]+gammau1[a1,b1,c1]*eps; od; od;
od;

for a1 from 1 to 4 do for b1 from 1 to 4 do
   for c1 from 1 to 4 do for d1 from 1 to 4 do
Riemann[a1,b1,c1,d1]:= diff(gammau[a1,b1,d1],x[c1]) -
diff(gammau[a1,b1,c1],x[d1])+
sum(gammau[a1,e1,c1]*gammau[e1,b1,d1]-gammau[a1,e1,d1]*gammau[e1,b1,c1],e1=1..4);
od; od; od; od;
```

```
for a1 from 1 to 4 do for b1 from 1 to 4 do
Ricci[a1,b1]:=simplify(sum(Riemann[e1,a1,e1,b1],e1=1..4));
Ricci0[a1,b1]:=simplify(subs(eps=0,Ricci[a1,b1]));
Ricci1[a1,b1]:=factor(simplify(subs(eps=0,diff(Ricci[a1,b1],eps))));
od; od;

interface(echo=4); Ricci1[1,1]; Ricci1[1,2]; Ricci1[1,3];
Ricci1[1,4]; Ricci1[2,3]; Ricci1[2,4]; Ricci1[3,4]; Ricci1[4,4];
factor(simplify(Ricci1[2,2]+Ricci1[3,3]/sin(th)^2));
factor(simplify(Ricci1[2,2]-Ricci1[3,3]/sin(th)^2));
```

B.2 program results

```
> read "t.map";
```

```
Warning, the protected names norm and trace have been redefined and
unprotected
```

$$gd := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$gu := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$x := [r,\ th,\ ph,\ t]$$

$$gammad := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ [])$$

$$gammau := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ [])$$

$$gammaut := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ [])$$

$$gammau0 := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ [])$$

$$gammau1 := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ [])$$

$$Riemann := \mathrm{array}(1..4,\ 1..4,\ 1..4,\ 1..4,\ [])$$

$$Ricci := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$Ricci0 := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$Ricci1 := \mathrm{array}(1..4,\ 1..4,\ [])$$

$$P2 := \frac{3}{2}\cos(th)^2 - \frac{1}{2}$$

$$P2\_t := -3\cos(th)\sin(th)$$

$$P2\_tt := 3\sin(th)^2 - 3\cos(th)^2$$

$$\left[\frac{1 + 2\, eps\, e^{(I\, s\, t)}\, (\frac{3}{2}\cos(th)^2 - \frac{1}{2})\, \mathrm{L}(r)}{1 - \frac{2\, M}{r}},\ 0,\ 0,\ 0\right]$$

$$\left[0,\ r^2\, (1 + 2\, eps\, e^{(I\, s\, t)}\, ((\frac{3}{2}\cos(th)^2 - \frac{1}{2})\, \mathrm{T}(r) + (3\sin(th)^2 - 3\cos(th)^2)\, \mathrm{V}(r))),\ 0,\ 0\right]$$

$$\left[0,\ 0,\right.$$

$$r^2\sin(th)^2\, (1 + 2\, eps\, e^{(I\, s\, t)}\, ((\frac{3}{2}\cos(th)^2 - \frac{1}{2})\, \mathrm{T}(r) - 3\cos(th)\sin(th)\cot(th)\, \mathrm{V}(r))),$$

$$\left.0\right]$$

$$\left[0,\ 0,\ 0,\ -(1 - \frac{2\, m}{r})\, (1 + eps\, e^{(I\, s\, t)}\, (3\cos(th)^2 - 1)\, \mathrm{N}(r))\right]$$

$$gu :=$$

$$\left[-\frac{-r + 2\, M}{(1 + 3\, eps\, e^{(I\, s\, t)}\, \mathrm{L}(r)\cos(th)^2 - eps\, e^{(I\, s\, t)}\, \mathrm{L}(r))\, r},\ 0,\ 0,\ 0\right]$$

$$\left[0, -1/((-1 - 3\, eps\, e^{(I\, s\, t)}\, \mathrm{T}(r)\cos(th)^2 + eps\, e^{(I\, s\, t)}\, \mathrm{T}(r) - 6\, eps\, e^{(I\, s\, t)}\, \mathrm{V}(r)\sin(th)^2\right.$$

$$\left. + 6\, eps\, e^{(I\, s\, t)}\, \mathrm{V}(r)\cos(th)^2) r^2),\ 0,\ 0\right]$$

$$\left[0,\ 0, 1/((1 + 3\, eps\, e^{(I\, s\, t)}\, \mathrm{T}(r)\cos(th)^2 - eps\, e^{(I\, s\, t)}\, \mathrm{T}(r)\right.$$

$$\left. - 6\, eps\, e^{(I\, s\, t)}\cos(th)\sin(th)\cot(th)\, \mathrm{V}(r))\sin(th)^2\, r^2),\ 0\right]$$

$$\left[0,\ 0,\ 0,\ \frac{r}{(1 + 3\, eps\, e^{(I\, s\, t)}\, \mathrm{N}(r)\cos(th)^2 - eps\, e^{(I\, s\, t)}\, \mathrm{N}(r))\, (-r + 2\, M)}\right]$$

```
> Ricci1[1,1];
```

$$\frac{1}{2} e^{(I\,s\,t)}\,(3\cos(th)^2-1)(-12\,M\,\mathrm{L}(r)+12\,r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))-(\tfrac{\partial^2}{\partial r^2}\,\mathrm{N}(r))\,r^3+6\,r\,\mathrm{L}(r)$$
$$-\,2\,r^3\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{T}(r))+6\,r^3\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r))+6\,(\tfrac{\partial}{\partial r}\,\mathrm{L}(r))\,M^2+6\,(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,M^2+2\,(\tfrac{\partial}{\partial r}\,\mathrm{L}(r))\,r^2$$
$$+\,36\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))\,M^2-4\,r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))-12\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))\,M^2+24\,r\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r))\,M^2$$
$$+\,8\,r^2\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{T}(r))\,M+4\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{N}(r))\,r^2\,M-3\,(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,r\,M-7\,(\tfrac{\partial}{\partial r}\,\mathrm{L}(r))\,r\,M$$
$$-\,42\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))+14\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))-8\,r\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{T}(r))\,M^2-r^3\,s^2\,\mathrm{L}(r)$$
$$-\,4\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{N}(r))\,r\,M^2-24\,r^2\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r))\,M)\Big/((-r+2\,M)^2\,r)$$

```
> Ricci1[1,2];
```

$$-3\sin(th)\cos(th)\,e^{(I\,s\,t)}(3\,M\,\mathrm{N}(r)+(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,r^2+r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))-r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))-r\,\mathrm{L}(r)$$
$$+\,M\,\mathrm{L}(r)-2\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))+2\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))-2\,(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,r\,M-\mathrm{N}(r)\,r)/($$
$$r\,(-r+2\,M))$$

```
> Ricci1[1,3];
```

$$0$$

```
> Ricci1[1,4];
```

$$I\,s\,e^{(I\,s\,t)}\,(3\cos(th)^2-1)(r\,\mathrm{T}(r)+r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))-2\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))+6\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))$$
$$-\,3\,M\,\mathrm{T}(r)+9\,M\,\mathrm{V}(r)-3\,r\,\mathrm{V}(r)-3\,r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))-r\,\mathrm{L}(r)+2\,M\,\mathrm{L}(r))/($$
$$r\,(-r+2\,M))$$

```
> Ricci1[2,3];
```

$$0$$

```
> Ricci1[2,4];
```

$$3\,I\sin(th)\,(\mathrm{L}(r)-\mathrm{V}(r)+\mathrm{T}(r))\,s\,e^{(I\,s\,t)}\cos(th)$$

```
> Ricci1[3,4];
```

$$0$$

```
> Ricci1[4,4];
```

$$\frac{1}{2}e^{(I\,s\,t)}\,(3\cos(th)^2-1)(-6\,\mathrm{N}(r)\,r+2\,(\tfrac{\partial}{\partial r}\,\mathrm{L}(r))\,M^2-4\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))\,M^2+12\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))\,M^2$$
$$-\,4\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{N}(r))\,r^2\,M-5\,(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,r\,M+2\,(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,m^2+(\tfrac{\partial^2}{\partial r^2}\,\mathrm{N}(r))\,r^3$$
$$+\,2\,m\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))+12\,M\,\mathrm{N}(r)-(\tfrac{\partial}{\partial r}\,\mathrm{L}(r))\,r\,M-6\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))+2\,(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,r^2$$
$$+\,r^3\,s^2\,\mathrm{L}(r)-6\,r^3\,s^2\,\mathrm{V}(r)+2\,r^3\,s^2\,\mathrm{T}(r)+4\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{N}(r))\,r\,M^2)/r^3$$

```
> factor(simplify(Ricci1[2,2]+Ricci1[3,3]/sin(th)^2));
```

$$((3\cos(th)^2-1)(-4\,r^2\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{T}(r))\,M-12\,r\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r))\,M^2+12\,r^2\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r))\,M$$
$$+\,4\,r\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{T}(r))\,M^2-4\,(\tfrac{\partial}{\partial r}\,\mathrm{L}(r))\,M^2+4\,(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,M^2+12\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))\,M^2$$
$$-\,36\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))\,M^2-(\tfrac{\partial}{\partial r}\,\mathrm{L}(r))\,r^2+6\,M\,\mathrm{N}(r)+(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,r^2-4\,r\,\mathrm{T}(r)$$
$$+\,4\,r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))-12\,r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))+8\,M\,\mathrm{T}(r)-5\,r\,\mathrm{L}(r)+10\,M\,\mathrm{L}(r)$$
$$-\,14\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{T}(r))+42\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))-4\,(\tfrac{\partial}{\partial r}\,\mathrm{N}(r))\,r\,M-3\,\mathrm{N}(r)\,r+r^3\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{T}(r))$$
$$-\,3\,r^3\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r))+4\,(\tfrac{\partial}{\partial r}\,\mathrm{L}(r))\,r\,M+r^3\,s^2\,\mathrm{T}(r)-3\,r^3\,s^2\,\mathrm{V}(r))e^{(I\,s\,t)})/(-r+2\,M)$$

```
> factor(simplify(Ricci1[2,2]-Ricci1[3,3]/sin(th)^2));
```

$$-3((\cos(th)-1)\,(\cos(th)+1)(-6\,M\,r\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))+r^3\,s^2\,\mathrm{V}(r)+4\,r\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r))\,M^2$$
$$-\,4\,r^2\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r))\,M+4\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))\,M^2+r\,\mathrm{L}(r)-2\,M\,\mathrm{L}(r)+\mathrm{N}(r)\,r-2\,M\,\mathrm{N}(r)$$
$$+\,2\,r^2\,(\tfrac{\partial}{\partial r}\,\mathrm{V}(r))+r^3\,(\tfrac{\partial^2}{\partial r^2}\,\mathrm{V}(r)))e^{(I\,s\,t)})/(-r+2\,M)$$

Appendix C

Maple computer program 3

This program transforms even-parity metric perturbations of a Schwarzschild black hole to Bondi-Sachs form. See chapter 5 sec. 5.3

```
> a1:=-(1-2*M/r)*dt^2+(1-2*M/r)^(-1)*dr^2+r^2*(d(theta)^2+(sin(theta))^
> 2*d(phi)^2);
```

$$a1 := -(1 - \frac{2\,M}{r})\,dt^2 + \frac{dr^2}{1 - \frac{2\,M}{r}} + r^2\,(\mathrm{d}(\theta)^2 + \sin(\theta)^2\,\mathrm{d}(\phi)^2)$$

```
> a2:=dt^2=du^2-2*diff(F(r),r)*du*dr+diff(F(r),r)^2*dr^2;
```

$$a2 := dt^2 = du^2 - 2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr + (\tfrac{\partial}{\partial r}\,\mathrm{F}(r))^2\,dr^2$$

```
> a3:=subs(a2,a1);
```

$$a3 := -(1 - \frac{2\,M}{r})\,(du^2 - 2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr + (\tfrac{\partial}{\partial r}\,\mathrm{F}(r))^2\,dr^2) + \frac{dr^2}{1 - \frac{2\,M}{r}}$$
$$+ r^2\,(\mathrm{d}(\theta)^2 + \sin(\theta)^2\,\mathrm{d}(\phi)^2)$$

```
> a4:=(1-2*M/r)^(-1)-(1-2*M/r)*diff(F(r),r)^2=0;
```

$$a4 := \frac{1}{1 - \frac{2\,M}{r}} - (1 - \frac{2\,M}{r})\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))^2 = 0$$

```
> a5:=dsolve(a4);
```

$$a5 := \mathrm{F}(r) = -r - 2\,M\ln(r - 2\,M) + \_C1,\ \mathrm{F}(r) = r + 2\,M\ln(r - 2\,M) + \_C1$$

```
> b1:=-g_t*(1+2*epsilon*E*P_2(theta)*N(r))*dt^2+g_r*(1+2*epsilon*E*P_2(
> theta)*L(r))*dr^2+r^2*(1+2*epsilon*E*(P_2(theta)*T(r)+(3*sin(theta)^2-
> 3*cos(theta)^2)*V(r)))*d(theta)^2+r^2*sin(theta)^2*(1+2*epsilon*E*((P_
> 2(theta)*T(r)-3*cos(theta)*sin(theta)*cot(theta)*V(r))))*d(phi)^2;
```

$$b1 := -g\_t\,(1+2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r))\,dt^2 + g\_r\,(1+2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{L}(r))\,dr^2$$
$$+\,r^2\,(1+2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r)+(3\sin(\theta)^2-3\cos(\theta)^2)\,\mathrm{V}(r)))\,\mathrm{d}(\theta)^2$$
$$+\,r^2\sin(\theta)^2\,(1+2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r)-3\cos(\theta)\sin(\theta)\cot(\theta)\,\mathrm{V}(r)))\,\mathrm{d}(\phi)^2$$

```
> b2:=dt^2=(du+epsilon*f(r)*i*sigma*E*P_2(theta)*du+diff(F(r),r)*dr+eps
> ilon*diff(F(r),r)*f(r)*i*sigma*E*P_2(theta)*dr+epsilon*diff(f(r),r)*E*
> P_2(theta)*dr+epsilon*f(r)*E*diff(P_2(theta),theta)*d(theta))^2;
```

$$b2 := dt^2 = (du+\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du+(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr+\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,dr$$
$$+\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,dr+\varepsilon\,\mathrm{f}(r)\,E\,(\tfrac{\partial}{\partial \theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta))^2$$

```
> b3:=expand(b2);
```

$$b3 := dt^2 = 2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr+(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))^2\,dr^2+2\,\varepsilon^2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,\mathrm{f}(r)^2\,i\,\sigma\,E^2\,\mathrm{P\_2}(\theta)\,dr\,\%1\,\mathrm{d}(\theta)$$
$$+\,2\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,dr\,du+\varepsilon^2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))^2\,\mathrm{f}(r)^2\,i^2\,\sigma^2\,E^2\,\mathrm{P\_2}(\theta)^2\,dr^2$$
$$+\,2\,\varepsilon^2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,\mathrm{f}(r)\,i\,\sigma\,E^2\,\mathrm{P\_2}(\theta)^2\,dr^2\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))$$
$$+\,2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr^2\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)+2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,\%1\,\mathrm{d}(\theta)$$
$$+\,\varepsilon^2\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))^2\,E^2\,\mathrm{P\_2}(\theta)^2\,dr^2+2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))^2\,dr^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)$$
$$+\,4\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr+2\,\varepsilon^2\,\mathrm{f}(r)^2\,i^2\,\sigma^2\,E^2\,\mathrm{P\_2}(\theta)^2\,du\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr$$
$$+\,2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du^2+\varepsilon^2\,\mathrm{f}(r)^2\,i^2\,\sigma^2\,E^2\,\mathrm{P\_2}(\theta)^2\,du^2$$
$$+\,2\,\varepsilon^2\,\mathrm{f}(r)\,i\,\sigma\,E^2\,\mathrm{P\_2}(\theta)^2\,du\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,dr+2\,\varepsilon^2\,\mathrm{f}(r)^2\,i\,\sigma\,E^2\,\mathrm{P\_2}(\theta)\,du\,\%1\,\mathrm{d}(\theta)$$
$$+\,\varepsilon^2\,\mathrm{f}(r)^2\,E^2\,\%1^2\,\mathrm{d}(\theta)^2+2\,\varepsilon\,\mathrm{f}(r)\,E\,\%1\,\mathrm{d}(\theta)\,du$$
$$+\,2\,\varepsilon^2\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,E^2\,\mathrm{P\_2}(\theta)\,dr\,\mathrm{f}(r)\,\%1\,\mathrm{d}(\theta)+du^2$$
$$\%1 := \tfrac{\partial}{\partial \theta}\,\mathrm{P\_2}(\theta)$$

```
> b4:=subs(epsilon^2=0,b3);
```

$$b4 := dt^2 = 2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr+(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))^2\,dr^2+2\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,dr\,du$$
$$+\,2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr^2\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)+2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\tfrac{\partial}{\partial \theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)$$
$$+\,2\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))^2\,dr^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)+4\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du\,(\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr$$
$$+\,2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du^2+2\,\varepsilon\,\mathrm{f}(r)\,E\,(\tfrac{\partial}{\partial \theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du+du^2$$

```
> b5:=subs(b4,b1);
```

$$b5 := -g\_t\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r))(2\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr + (\frac{\partial}{\partial r}\,\mathrm{F}(r))^2\,dr^2$$
$$+ 2\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,dr\,du + 2\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,dr^2\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)$$
$$+ 2\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial \theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta) + 2\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))^2\,dr^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)$$
$$+ 4\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,dr + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du^2$$
$$+ 2\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial \theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du + du^2) + g\_r\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{L}(r))\,dr^2$$
$$+ r^2\,(1 + 2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) + (3\sin(\theta)^2 - 3\cos(\theta)^2)\,\mathrm{V}(r)))\,\mathrm{d}(\theta)^2$$
$$+ r^2\sin(\theta)^2\,(1 + 2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 3\cos(\theta)\sin(\theta)\cot(\theta)\,\mathrm{V}(r)))\,\mathrm{d}(\phi)^2$$

```
> b6:=subs(d(theta)^2=0,dr^2=0,du*dr=0,dr=0,d(theta)=0,d(phi)=0,b5);
```

$$b6 := -g\_t\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r))\,(2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du^2 + du^2)$$

```
> b7:=factor(b6);
```

$$b7 := -g\_t\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r))\,du^2\,(2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) + 1)$$

```
> b8:=subs(du^2=0,dr=0,d(theta)^2=0,d(phi)=0,b5);
```

$$b8 := -2\,g\_t\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r))\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial \theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du$$

```
> b9:=subs(du=0,dr^2=0,d(theta)^2=0,d(phi)=0,b5);
```

$$b9 := -2\,g\_t\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r))\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial \theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)$$

```
> b10:=subs(du^2=0,dr^2=0,d(theta)=0,d(phi)=0,b5);
```

$$b10 := -g\_t\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r))(2\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr + 2\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,dr\,du$$
$$+ 4\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,du\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,dr)$$

```
> b11:=factor(b10);
```

$$b11 := -2g\_t\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r))\,du\,dr$$
$$((\frac{\partial}{\partial r}\,\mathrm{F}(r)) + \varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta) + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,(\frac{\partial}{\partial r}\,\mathrm{F}(r)))$$

```
> b12:=subs(du^2=0,dr=0,du=0,d(phi)=0,b5);
```

$$b12 := r^2\,(1 + 2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) + (3\sin(\theta)^2 - 3\cos(\theta)^2)\,\mathrm{V}(r)))\,\mathrm{d}(\theta)^2$$

```
> b14:=factor(b13);
```

$$b14 := b13$$

```
> b15:=subs(d(theta)=0,dr=0,du=0,b5);
```

$$b15 := r^2\sin(\theta)^2\,(1 + 2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 3\cos(\theta)\sin(\theta)\cot(\theta)\,\mathrm{V}(r)))\,\mathrm{d}(\phi)^2$$

```
> b16:=b7+b8+b9+b11+b12+b15;
```

$$b16 := -g\_t\,\%1\,du^2\,(2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)+1) - 2\,g\_t\,\%1\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du$$
$$-\,2\,g\_t\,\%1\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)$$
$$-\,2\,g\_t\,\%1\,du\,dr\,((\frac{\partial}{\partial r}\,\mathrm{F}(r)) + \varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta) + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,(\frac{\partial}{\partial r}\,\mathrm{F}(r)))$$
$$+\,r^2\,(1+2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) + (3\sin(\theta)^2 - 3\cos(\theta)^2)\,\mathrm{V}(r)))\,\mathrm{d}(\theta)^2$$
$$+\,r^2\sin(\theta)^2\,(1+2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 3\cos(\theta)\sin(\theta)\cot(\theta)\,\mathrm{V}(r)))\,\mathrm{d}(\phi)^2$$
$$\%1 := 1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)$$

```
> b17:=subs(epsilon^2=0,b16);
```

$$b17 := -g\_t\,\%1\,du^2\,(2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)+1) - 2\,g\_t\,\%1\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du$$
$$-\,2\,g\_t\,\%1\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)$$
$$-\,2\,g\_t\,\%1\,du\,dr\,((\frac{\partial}{\partial r}\,\mathrm{F}(r)) + \varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta) + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,(\frac{\partial}{\partial r}\,\mathrm{F}(r)))$$
$$+\,r^2\,(1+2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) + (3\sin(\theta)^2 - 3\cos(\theta)^2)\,\mathrm{V}(r)))\,\mathrm{d}(\theta)^2$$
$$+\,r^2\sin(\theta)^2\,(1+2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 3\cos(\theta)\sin(\theta)\cot(\theta)\,\mathrm{V}(r)))\,\mathrm{d}(\phi)^2$$
$$\%1 := 1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)$$

```
> b18:=-g_t*diff(F(r),r)^2+2*g_t*diff(F(r),r)^2*epsilon*f(r)*i*sigma*E*
> P_2(theta)-2*g_t*diff(F(r),r)*epsilon*diff(f(r),r)*E*P_2(theta)-2*g_t*
> epsilon*E*P_2(theta)*N(r)*diff(F(r),r)^2+g_r+2*g_r*epsilon*E*P_2(theta
> )*L(r)=0;
```

$$b18 := -g\_t\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))^2 + 2\,g\_t\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)$$
$$-\,2\,g\_t\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta) - 2\,g\_t\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))^2 + g\_r$$
$$+\,2\,g\_r\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{L}(r) = 0$$

```
> b19:=subs(F(r)=r+2*M*ln(r-2*M),b18);
```

$$b19 := -g\_t\,\%1^2 + 2\,g\_t\,\%1^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) - 2\,g\_t\,\%1\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)$$
$$-\,2\,g\_t\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)\,\%1^2 + g\_r + 2\,g\_r\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{L}(r) = 0$$
$$\%1 := \frac{\partial}{\partial r}\,(r + 2\,M\ln(r - 2\,M))$$

```
> b20:=subs(g_t=(1-2*M/r),g_r=(1-2*M/r)^(-1),E=e^(i*sigma*t),epsilon^2=
> 0,b19);
```

$$b20 := -(1-\frac{2\,M}{r})\,(1+\frac{2\,M}{r-2\,M})^2 + 2\,(1-\frac{2\,M}{r})\,(1+\frac{2\,M}{r-2\,M})^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,e^{(i\,\sigma\,t)}\,\mathrm{P\_2}(\theta)$$
$$-2\,(1-\frac{2\,M}{r})\,(1+\frac{2\,M}{r-2\,M})\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,e^{(i\,\sigma\,t)}\,\mathrm{P\_2}(\theta)$$
$$-2\,(1-\frac{2\,M}{r})\,\varepsilon\,e^{(i\,\sigma\,t)}\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)\,(1+\frac{2\,M}{r-2\,M})^2 + \frac{1}{1-\frac{2\,M}{r}}$$
$$+\frac{2\,\varepsilon\,e^{(i\,\sigma\,t)}\,\mathrm{P\_2}(\theta)\,\mathrm{L}(r)}{1-\frac{2\,M}{r}} = 0$$

```
> b21:=simplify(b20);
```

$$b21 := -2\,\frac{\varepsilon\,e^{(i\,\sigma\,t)}\,\mathrm{P\_2}(\theta)\,(r\,\mathrm{f}(r)\,i\,\sigma - (\frac{\partial}{\partial r}\,\mathrm{f}(r))\,r + 2\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,M - r\,\mathrm{N}(r) + r\,\mathrm{L}(r))}{-r+2\,M} = 0$$

```
> b22:=dsolve(b21,f(r));
```

$$b22 := \mathrm{f}(r) = \left(\int \frac{r\,e^{(-r\,i\,\sigma)}\,(\mathrm{N}(r)-\mathrm{L}(r))\,(r-2\,M)^{(-2\,i\,\sigma\,M)}}{-r+2\,M}\,dr + \_C1\right)\,e^{(r\,i\,\sigma)}\,((r-2\,M)^{(i\,\sigma\,M)})^2$$

```
> b23:=f(r)=e^(r*i*sigma/(r-2*M))*(int(e^(-r*i*sigma/(-r+2*M))*((N(r)*r
> -r*L(r))/(-r+2*M)),r));
```

$$b23 := \mathrm{f}(r) = e^{(\frac{r\,i\,\sigma}{r-2\,M})}\int \frac{e^{(-\frac{r\,i\,\sigma}{-r+2\,M})}\,(r\,\mathrm{N}(r) - r\,\mathrm{L}(r))}{-r+2\,M}\,dr$$

```
> b24:=subs(F(r)=r+2*M*ln(r-2*M),b17);
```

$$b24 := -g\_t\,\%1\,du^2\,(2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)+1) - 2\,g\_t\,\%1\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du$$
$$-2\,g\_t\,\%1\,(\frac{\partial}{\partial r}\,(r+2\,M\ln(r-2\,M)))\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta) - 2\,g\_t\,\%1\,du$$
$$dr((\frac{\partial}{\partial r}\,(r+2\,M\ln(r-2\,M))) + \varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)$$
$$+2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,(\frac{\partial}{\partial r}\,(r+2\,M\ln(r-2\,M))))$$
$$+r^2\,(1+2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) + (3\sin(\theta)^2 - 3\cos(\theta)^2)\,\mathrm{V}(r)))\,\mathrm{d}(\theta)^2$$
$$+r^2\sin(\theta)^2\,(1+2\,\varepsilon\,E\,(\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 3\cos(\theta)\sin(\theta)\cot(\theta)\,\mathrm{V}(r)))\,\mathrm{d}(\phi)^2$$
$$\%1 := 1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)$$

```
> b25:=expand(b24);
```

$$
\begin{aligned}
b25 :=& -4\,\frac{g\_t\; dr\; \varepsilon\, \mathrm{f}(r)\, E\, \%1\, \mathrm{d}(\theta)\, M}{r-2\,M} - 4\, g\_t\; dr\; \varepsilon^2\, \mathrm{f}(r)\, E^2\, \%1\, \mathrm{d}(\theta)\, \mathrm{P\_2}(\theta)\, \mathrm{N}(r) \\
&- 4\, g\_t\; \varepsilon^2\, \mathrm{f}(r)\, E^2\, \%1\, \mathrm{d}(\theta)\; du\; \mathrm{P\_2}(\theta)\, \mathrm{N}(r) - 2\, g\_t\; dr\; \varepsilon\, \mathrm{f}(r)\, E\, \%1\, \mathrm{d}(\theta) \\
&- 2\, g\_t\; \varepsilon\, \mathrm{f}(r)\, E\, \%1\, \mathrm{d}(\theta)\; du - \frac{8\, g\_t\; dr\; \varepsilon^2\, \mathrm{f}(r)\, E^2\, \%1\, \mathrm{d}(\theta)\, \mathrm{P\_2}(\theta)\, \mathrm{N}(r)\, M}{r-2\,M} \\
&- 2\, g\_t\; du\; dr - 2\, g\_t\; du^2\, \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{N}(r) - 2\, g\_t\; du^2\, \varepsilon\, \mathrm{f}(r)\, i\, \sigma\, E\, \mathrm{P\_2}(\theta) \\
&- 4\, g\_t\; du^2\, \varepsilon^2\, E^2\, \mathrm{P\_2}(\theta)^2\, \mathrm{N}(r)\, \mathrm{f}(r)\, i\, \sigma + 2\, r^2\, \mathrm{d}(\theta)^2\, \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{T}(r) \\
&+ 6\, r^2\, \mathrm{d}(\theta)^2\, \varepsilon\, E\, \mathrm{V}(r)\, \sin(\theta)^2 - 6\, r^2\, \mathrm{d}(\theta)^2\, \varepsilon\, E\, \mathrm{V}(r)\, \cos(\theta)^2 \\
&- 8\, g\_t\; du\; dr\; \varepsilon^2\, E^2\, \mathrm{P\_2}(\theta)^2\, \mathrm{N}(r)\, \mathrm{f}(r)\, i\, \sigma \\
&- \frac{16\, g\_t\; du\; dr\; \varepsilon^2\, E^2\, \mathrm{P\_2}(\theta)^2\, \mathrm{N}(r)\, \mathrm{f}(r)\, i\, \sigma\, M}{r-2\,M} - \frac{8\, g\_t\; du\; dr\; \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{N}(r)\, M}{r-2\,M} \\
&- 4\, g\_t\; du\; dr\; \varepsilon^2\, E^2\, \mathrm{P\_2}(\theta)^2\, \mathrm{N}(r)\, (\tfrac{\partial}{\partial r}\, \mathrm{f}(r)) - \frac{8\, g\_t\; du\; dr\; \varepsilon\, \mathrm{f}(r)\, i\, \sigma\, E\, \mathrm{P\_2}(\theta)\, M}{r-2\,M} \\
&- 4\, g\_t\; du\; dr\; \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{N}(r) - 2\, g\_t\; du\; dr\; \varepsilon\, (\tfrac{\partial}{\partial r}\, \mathrm{f}(r))\, E\, \mathrm{P\_2}(\theta) \\
&- 4\, g\_t\; du\; dr\; \varepsilon\, \mathrm{f}(r)\, i\, \sigma\, E\, \mathrm{P\_2}(\theta) - \frac{4\, g\_t\; du\; dr\; M}{r-2\,M} \\
&+ 2\, r^2\, \sin(\theta)^2\, \mathrm{d}(\phi)^2\, \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{T}(r) - 6\, r^2\, \sin(\theta)^3\, \mathrm{d}(\phi)^2\, \varepsilon\, E\, \cos(\theta)\, \cot(\theta)\, \mathrm{V}(r) \\
&+ r^2\, \mathrm{d}(\theta)^2 - g\_t\; du^2 + r^2\, \sin(\theta)^2\, \mathrm{d}(\phi)^2 \\
\%1 :=& \tfrac{\partial}{\partial \theta}\, \mathrm{P\_2}(\theta)
\end{aligned}
$$

```
> b26:=subs(epsilon^2=0,b25);
```

$$
\begin{aligned}
b26 :=& -4\,\frac{g\_t\; dr\; \varepsilon\, \mathrm{f}(r)\, E\, (\frac{\partial}{\partial \theta}\, \mathrm{P\_2}(\theta))\, \mathrm{d}(\theta)\, M}{r-2\,M} - 2\, g\_t\; dr\; \varepsilon\, \mathrm{f}(r)\, E\, (\tfrac{\partial}{\partial \theta}\, \mathrm{P\_2}(\theta))\, \mathrm{d}(\theta) \\
&- 2\, g\_t\; \varepsilon\, \mathrm{f}(r)\, E\, (\tfrac{\partial}{\partial \theta}\, \mathrm{P\_2}(\theta))\, \mathrm{d}(\theta)\; du - 2\, g\_t\; du\; dr - 2\, g\_t\; du^2\, \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{N}(r) \\
&- 2\, g\_t\; du^2\, \varepsilon\, \mathrm{f}(r)\, i\, \sigma\, E\, \mathrm{P\_2}(\theta) + 2\, r^2\, \mathrm{d}(\theta)^2\, \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{T}(r) \\
&+ 6\, r^2\, \mathrm{d}(\theta)^2\, \varepsilon\, E\, \mathrm{V}(r)\, \sin(\theta)^2 - 6\, r^2\, \mathrm{d}(\theta)^2\, \varepsilon\, E\, \mathrm{V}(r)\, \cos(\theta)^2 \\
&- \frac{8\, g\_t\; du\; dr\; \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{N}(r)\, M}{r-2\,M} - \frac{8\, g\_t\; du\; dr\; \varepsilon\, \mathrm{f}(r)\, i\, \sigma\, E\, \mathrm{P\_2}(\theta)\, M}{r-2\,M} \\
&- 4\, g\_t\; du\; dr\; \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{N}(r) - 2\, g\_t\; du\; dr\; \varepsilon\, (\tfrac{\partial}{\partial r}\, \mathrm{f}(r))\, E\, \mathrm{P\_2}(\theta) \\
&- 4\, g\_t\; du\; dr\; \varepsilon\, \mathrm{f}(r)\, i\, \sigma\, E\, \mathrm{P\_2}(\theta) - \frac{4\, g\_t\; du\; dr\; M}{r-2\,M} \\
&+ 2\, r^2\, \sin(\theta)^2\, \mathrm{d}(\phi)^2\, \varepsilon\, E\, \mathrm{P\_2}(\theta)\, \mathrm{T}(r) - 6\, r^2\, \sin(\theta)^3\, \mathrm{d}(\phi)^2\, \varepsilon\, E\, \cos(\theta)\, \cot(\theta)\, \mathrm{V}(r) \\
&+ r^2\, \mathrm{d}(\theta)^2 - g\_t\; du^2 + r^2\, \sin(\theta)^2\, \mathrm{d}(\phi)^2
\end{aligned}
$$

```
> b27:=expand(b26);
```

$$b27 := -4\,\frac{g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,M}{r-2\,M} - 2\,g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)$$
$$- 2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du - 2\,g\_t\,du\,dr - 2\,g\_t\,du^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)$$
$$- 2\,g\_t\,du^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) + 2\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r)$$
$$+ 6\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{V}(r)\sin(\theta)^2 - 6\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{V}(r)\cos(\theta)^2$$
$$- \frac{8\,g\_t\,du\,dr\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)\,M}{r-2\,M} - \frac{8\,g\_t\,du\,dr\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,M}{r-2\,M}$$
$$- 4\,g\_t\,du\,dr\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r) - 2\,g\_t\,du\,dr\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)$$
$$- 4\,g\_t\,du\,dr\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) - \frac{4\,g\_t\,du\,dr\,M}{r-2\,M}$$
$$+ 2\,r^2\sin(\theta)^2\,\mathrm{d}(\phi)^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 6\,r^2\sin(\theta)^3\,\mathrm{d}(\phi)^2\,\varepsilon\,E\cos(\theta)\cot(\theta)\,\mathrm{V}(r)$$
$$+ r^2\,\mathrm{d}(\theta)^2 - g\_t\,du^2 + r^2\sin(\theta)^2\,\mathrm{d}(\phi)^2$$

```
> b28:=subs(epsilon^2=0,b27);
```

$$b28 := -4\,\frac{g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,M}{r-2\,M} - 2\,g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)$$
$$- 2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du - 2\,g\_t\,du\,dr - 2\,g\_t\,du^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)$$
$$- 2\,g\_t\,du^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) + 2\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r)$$
$$+ 6\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{V}(r)\sin(\theta)^2 - 6\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{V}(r)\cos(\theta)^2$$
$$- \frac{8\,g\_t\,du\,dr\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)\,M}{r-2\,M} - \frac{8\,g\_t\,du\,dr\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,M}{r-2\,M}$$
$$- 4\,g\_t\,du\,dr\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r) - 2\,g\_t\,du\,dr\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)$$
$$- 4\,g\_t\,du\,dr\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) - \frac{4\,g\_t\,du\,dr\,M}{r-2\,M}$$
$$+ 2\,r^2\sin(\theta)^2\,\mathrm{d}(\phi)^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 6\,r^2\sin(\theta)^3\,\mathrm{d}(\phi)^2\,\varepsilon\,E\cos(\theta)\cot(\theta)\,\mathrm{V}(r)$$
$$+ r^2\,\mathrm{d}(\theta)^2 - g\_t\,du^2 + r^2\sin(\theta)^2\,\mathrm{d}(\phi)^2$$

```
> b29:=subs(d(theta)^2=0,dr^2=0,du*dr=0,dr=0,d(theta)=0,d(phi)=0,b28);
```

$$b29 := -2\,g\_t\,du^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r) - 2\,g\_t\,du^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) - g\_t\,du^2$$

```
> b30:=factor(b29);
```

$$b30 := -g\_t\,du^2\,(2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r) + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) + 1)$$

```
> b31:=subs(du^2=0,dr=0,d(theta)^2=0,d(phi)=0,b28);
```

$$b31 := -2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du$$

```
> b32:=subs(du=0,dr^2=0,d(theta)^2=0,d(phi)=0,b28);
```

$$b32 := -4\,\frac{g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,M}{r-2\,M} - 2\,g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)$$

```
> b33:=factor(b32);
```

$$b33 := 2\,\frac{g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,r}{-r+2\,M}$$

```
> b34:=subs(du^2=0,dr^2=0,d(theta)=0,d(phi)=0,b28);
```

$$b34 := -2\,g\_t\,du\,dr - \frac{8\,g\_t\,du\,dr\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)\,M}{r-2\,M} - \frac{8\,g\_t\,du\,dr\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,M}{r-2\,M}$$
$$-4\,g\_t\,du\,dr\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r) - 2\,g\_t\,du\,dr\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)$$
$$-4\,g\_t\,du\,dr\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) - \frac{4\,g\_t\,du\,dr\,M}{r-2\,M}$$

```
> b35:=factor(b34);
```

$$b35 := 2g\_t\,du\,dr(r + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)\,r + \varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,r$$
$$-2\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,M + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,r)/(-r+2\,M)$$

```
> b36:=subs(du^2=0,dr=0,du=0,d(phi)=0,b28);
```

$$b36 := r^2\,\mathrm{d}(\theta)^2 + 2\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r) + 6\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{V}(r)\sin(\theta)^2$$
$$-6\,r^2\,\mathrm{d}(\theta)^2\,\varepsilon\,E\,\mathrm{V}(r)\cos(\theta)^2$$

```
> b37:=factor(b36);
```

$$b37 := r^2\,\mathrm{d}(\theta)^2\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r) + 6\,\varepsilon\,E\,\mathrm{V}(r)\sin(\theta)^2 - 6\,\varepsilon\,E\,\mathrm{V}(r)\cos(\theta)^2)$$

```
> b38:=subs(d(theta)=0,dr=0,du=0,b28);
```

$$b38 := r^2\sin(\theta)^2\,\mathrm{d}(\phi)^2 + 2\,r^2\sin(\theta)^2\,\mathrm{d}(\phi)^2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r)$$
$$-6\,r^2\sin(\theta)^3\,\mathrm{d}(\phi)^2\,\varepsilon\,E\cos(\theta)\cot(\theta)\,\mathrm{V}(r)$$

```
> b39:=factor(b38);
```

$$b39 := r^2\sin(\theta)^2\,\mathrm{d}(\phi)^2\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 6\,\varepsilon\,E\cos(\theta)\sin(\theta)\cot(\theta)\,\mathrm{V}(r))$$

```
> b40:=b30+b31+b33+b35+b37+b39;
```

$$b40 := -g\_t\,du^2\,(2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r) + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta) + 1)$$
$$-2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,du + \frac{2\,g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial\theta}\,\mathrm{P\_2}(\theta))\,\mathrm{d}(\theta)\,r}{-r+2\,M} + 2\,g\_t$$
$$du\,dr(r + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{N}(r)\,r + \varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,r - 2\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\theta)\,M$$
$$+ 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\theta)\,r)/(-r+2\,M)$$
$$+ r^2\,\mathrm{d}(\theta)^2\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r) + 6\,\varepsilon\,E\,\mathrm{V}(r)\sin(\theta)^2 - 6\,\varepsilon\,E\,\mathrm{V}(r)\cos(\theta)^2)$$
$$+ r^2\sin(\theta)^2\,\mathrm{d}(\phi)^2\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\theta)\,\mathrm{T}(r) - 6\,\varepsilon\,E\cos(\theta)\sin(\theta)\cot(\theta)\,\mathrm{V}(r))$$

```
> b41:=d(theta)=d(psi)+epsilon*diff(alpha(u,r,psi),u)*du+epsilon*diff(a
> lpha(u,r,psi),r)*dr+epsilon*diff(alpha(u,r,psi),psi)*d(psi);
```

$$b41 := \mathrm{d}(\theta) = \mathrm{d}(\psi) + \varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du + \varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr + \varepsilon\,(\tfrac{\partial}{\partial \psi}\,\alpha(u,\,r,\,\psi))\,\mathrm{d}(\psi)$$

```
> b42:=d(theta)^2=(d(psi)+epsilon*diff(alpha(u,r,psi),u)*du+epsilon*dif
> f(alpha(u,r,psi),r)*dr+epsilon*diff(alpha(u,r,psi),psi)*d(psi))^2;
```

$$b42 := \mathrm{d}(\theta)^2 = (\mathrm{d}(\psi) + \varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du + \varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr + \varepsilon\,(\tfrac{\partial}{\partial \psi}\,\alpha(u,\,r,\,\psi))\,\mathrm{d}(\psi))^2$$

```
> b43:=expand(b42);
```

$$b43 := \mathrm{d}(\theta)^2 = \mathrm{d}(\psi)^2 + 2\,\mathrm{d}(\psi)\,\varepsilon\,\%3\,du + 2\,\mathrm{d}(\psi)\,\varepsilon\,\%2\,dr + 2\,\varepsilon\,\%1\,\mathrm{d}(\psi)^2 + \varepsilon^2\,\%3^2\,du^2$$
$$+ 2\,\varepsilon^2\,\%3\,du\,\%2\,dr + 2\,\varepsilon^2\,\%3\,du\,\%1\,\mathrm{d}(\psi) + \varepsilon^2\,\%2^2\,dr^2 + 2\,\varepsilon^2\,\%2\,dr\,\%1\,\mathrm{d}(\psi)$$
$$+ \varepsilon^2\,\%1^2\,\mathrm{d}(\psi)^2$$
$$\%1 := \tfrac{\partial}{\partial \psi}\,\alpha(u,\,r,\,\psi)$$
$$\%2 := \tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi)$$
$$\%3 := \tfrac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi)$$

```
> b44:=subs(epsilon^2=0,b43);
```

$$b44 := \mathrm{d}(\theta)^2 = \mathrm{d}(\psi)^2 + 2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du + 2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr$$
$$+ 2\,\varepsilon\,(\tfrac{\partial}{\partial \psi}\,\alpha(u,\,r,\,\psi))\,\mathrm{d}(\psi)^2$$

```
> b45:=subs(b44,sin(theta)^2=sin(psi)^2*(1-2*epsilon*alpha*cot(psi)),co
> s(theta)^2=cos(psi)^2*(1-2*epsilon*alpha*cot(psi)),sin(theta)=sin(psi)
> *(1-2*epsilon*alpha*cot(psi))^(1/2),cos(theta)=cos(psi)*(1-2*epsilon*a
> lpha*cot(psi))^(1/2),diff(P_2(theta),theta)=diff(P_2(psi),psi),P_2(the
> ta)=P_2(psi),cot(theta)=cot(psi),d(theta)=d(psi),b40);
```

$$b45 := -g\_t\;du^2\,(2\,\varepsilon\,E\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r) + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\psi) + 1)$$
$$- 2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,E\,(\tfrac{\partial}{\partial \psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du + \frac{2\,g\_t\;dr\,\varepsilon\,\mathrm{f}(r)\,E\,(\frac{\partial}{\partial \psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r + 2\,M} + 2$$
$$g\_t\;du\;dr(r + 2\,\varepsilon\,E\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)\,r + \varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\psi)\,r$$
$$- 2\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,E\,\mathrm{P\_2}(\psi)\,M + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,E\,\mathrm{P\_2}(\psi)\,r)/(-r + 2\,M) + r^2(\mathrm{d}(\psi)^2$$
$$+ 2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du + 2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr$$
$$+ 2\,\varepsilon\,(\tfrac{\partial}{\partial \psi}\,\alpha(u,\,r,\,\psi))\,\mathrm{d}(\psi)^2)$$
$$(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) + 6\,\varepsilon\,E\,\mathrm{V}(r)\sin(\psi)^2\,\%1 - 6\,\varepsilon\,E\,\mathrm{V}(r)\cos(\psi)^2\,\%1)+$$
$$r^2\sin(\psi)^2\,\%1\,\mathrm{d}(\phi)^2\,(1 + 2\,\varepsilon\,E\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) - 6\,\varepsilon\,E\cos(\psi)\,\%1\sin(\psi)\cot(\psi)\,\mathrm{V}(r))$$
$$\%1 := 1 - 2\,\varepsilon\,\alpha\cot(\psi)$$

```
> b46:=subs(b44,E=e^(i*sigma*u+i*sigma*F(r)),b45);
```

$$b46 := -g\_t\,du^2\,(2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r) + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,\%2\,\mathrm{P\_2}(\psi) + 1)$$
$$-\,2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,\%2\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du + \frac{2\,g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,\%2\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M} +$$
$$2\,g\_t\,du\,dr(r + 2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)\,r + \varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,\%2\,\mathrm{P\_2}(\psi)\,r$$
$$-\,2\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,\%2\,\mathrm{P\_2}(\psi)\,M + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,\%2\,\mathrm{P\_2}(\psi)\,r)/(-r+2\,M) + r^2(\mathrm{d}(\psi)^2$$
$$+\,2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du + 2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr$$
$$+\,2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,r,\,\psi))\,\mathrm{d}(\psi)^2)$$
$$(1 + 2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) + 6\,\varepsilon\,\%2\,\mathrm{V}(r)\sin(\psi)^2\,\%1 - 6\,\varepsilon\,\%2\,\mathrm{V}(r)\cos(\psi)^2\,\%1) +$$
$$r^2\sin(\psi)^2\,\%1\,\mathrm{d}(\phi)^2$$
$$(1 + 2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) - 6\,\varepsilon\,\%2\cos(\psi)\,\%1\sin(\psi)\cot(\psi)\,\mathrm{V}(r))$$
$$\%1 := 1 - 2\,\varepsilon\,\alpha\cot(\psi)$$
$$\%2 := e^{(i\,\sigma\,u + i\,\sigma\,\mathrm{F}(r))}$$

```
> b47:=subs(d(phi)=0,d(psi)^2=0,du*dr=0,du^2=0,du=0,dr*dr=0,b46);
```

$$b47 := 2\,\frac{g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,\%1\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M} + 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr(1$$
$$+\,2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) + 6\,\varepsilon\,\%1\,\mathrm{V}(r)\sin(\psi)^2\,(1 - 2\,\varepsilon\,\alpha\cot(\psi))$$
$$-\,6\,\varepsilon\,\%1\,\mathrm{V}(r)\cos(\psi)^2\,(1 - 2\,\varepsilon\,\alpha\cot(\psi)))$$
$$\%1 := e^{(i\,\sigma\,u + i\,\sigma\,\mathrm{F}(r))}$$

```
> b48:=expand(b47);
```

$$b48 := 2\,\frac{g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M} + 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,\%1\,dr$$
$$+\,4\,r^2\,\mathrm{d}(\psi)\,\varepsilon^2\,\%1\,dr\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r)$$
$$+\,12\,r^2\,\mathrm{d}(\psi)\,\varepsilon^2\,\%1\,dr\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\sin(\psi)^2$$
$$-\,24\,r^2\,\mathrm{d}(\psi)\,\varepsilon^3\,\%1\,dr\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\sin(\psi)^2\,\alpha\cot(\psi)$$
$$-\,12\,r^2\,\mathrm{d}(\psi)\,\varepsilon^2\,\%1\,dr\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\cos(\psi)^2$$
$$+\,24\,r^2\,\mathrm{d}(\psi)\,\varepsilon^3\,\%1\,dr\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\cos(\psi)^2\,\alpha\cot(\psi)$$
$$\%1 := \tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi)$$

```
> b49:=subs(epsilon^2=0,g_t=(1-2*M/r),b48);
```

$$b49 := 2\,\frac{(1 - \frac{2\,M}{r})\,dr\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M}$$
$$+\,2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr$$
$$-\,24\,r^2\,\mathrm{d}(\psi)\,\varepsilon^3\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\sin(\psi)^2\,\alpha\cot(\psi)$$
$$+\,24\,r^2\,\mathrm{d}(\psi)\,\varepsilon^3\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\cos(\psi)^2\,\alpha\cot(\psi)$$

```
> b50:=subs(epsilon^2=0,simplify(b49));
```

$$b50 := 2\,\frac{dr\,\varepsilon\,\mathrm{d}(\psi)\,(-\mathrm{f}(r)\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\sin(\psi)+r^2\,(\frac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\sin(\psi))}{\sin(\psi)}$$

```
> b51:=lcoeff(b50,[dr,d(psi)]);
```

$$b51 := 2\,\frac{\varepsilon\,(-\mathrm{f}(r)\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\sin(\psi)+r^2\,(\frac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\sin(\psi))}{\sin(\psi)}$$

```
> b52:=dsolve(b51,alpha(u,r,psi));
```

$$b52 := \alpha(u,\,r,\,\psi)=\int\frac{\mathrm{f}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))}{r^2}\,dr+\_\mathrm{F1}(u,\,\psi)$$

```
> b53:=subs(b41,E=e^(i*sigma*u+i*sigma*F(r)),b45);
```

$$b53 := -g\_t\,du^2\,(2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)+2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,\%2\,\mathrm{P\_2}(\psi)+1)$$
$$-\,2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,\%2\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du+\frac{2\,g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,\%2\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M}+$$
$$2\,g\_t\,du\,dr(r+2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)\,r+\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,\%2\,\mathrm{P\_2}(\psi)\,r$$
$$-\,2\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,\%2\,\mathrm{P\_2}(\psi)\,M+2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,\%2\,\mathrm{P\_2}(\psi)\,r)/(-r+2\,M)+r^2(\mathrm{d}(\psi)^2$$
$$+\,2\,\mathrm{d}(\psi)\,\varepsilon\,(\frac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du+2\,\mathrm{d}(\psi)\,\varepsilon\,(\frac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr$$
$$+\,2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\alpha(u,\,r,\,\psi))\,\mathrm{d}(\psi)^2)$$
$$(1+2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r)+6\,\varepsilon\,\%2\,\mathrm{V}(r)\sin(\psi)^2\,\%1-6\,\varepsilon\,\%2\,\mathrm{V}(r)\cos(\psi)^2\,\%1)+$$
$$r^2\sin(\psi)^2\,\%1\,\mathrm{d}(\phi)^2$$
$$(1+2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r)-6\,\varepsilon\,\%2\cos(\psi)\,\%1\sin(\psi)\cot(\psi)\,\mathrm{V}(r))$$
$$\%1 := 1-2\,\varepsilon\,\alpha\cot(\psi)$$
$$\%2 := e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}$$

```
> b54:=subs(epsilon^2=0,epsilon^3=0,epsilon^4=0,epsilon^5=0,expand(b53)
> );
```

$$b54 := 2\,\frac{g_t\,du\,dr\,r}{-r+2\,M} + \frac{4\,g_t\,du\,dr\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,\mathrm{N}(r)\,r}{-r+2\,M}
+ \frac{2\,g_t\,du\,dr\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,r}{-r+2\,M}
- \frac{4\,g_t\,du\,dr\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,M}{-r+2\,M}
+ \frac{4\,g_t\,du\,dr\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,r}{-r+2\,M} - 2\,r^2\sin(\psi)^2\,\mathrm{d}(\phi)^2\,\varepsilon\,\alpha\cot(\psi)
+ 2\,r^2\sin(\psi)^2\,\mathrm{d}(\phi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,\mathrm{T}(r)
- 6\,r^2\sin(\psi)^3\,\mathrm{d}(\phi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\cos(\psi)\cot(\psi)\,\mathrm{V}(r)
+ 2\,r^2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\alpha(u,\,r,\,\psi))\,\mathrm{d}(\psi)^2 + 2\,r^2\,\mathrm{d}(\psi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,\mathrm{T}(r)
+ 6\,r^2\,\mathrm{d}(\psi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\sin(\psi)^2
- 6\,r^2\,\mathrm{d}(\psi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\cos(\psi)^2 + 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\frac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du
- 2\,g_t\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P_2}(\psi))\,\mathrm{d}(\psi)\,du
+ \frac{2\,g_t\,dr\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M}
+ 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\frac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr - 2\,g_t\,du^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,\mathrm{N}(r)
- 2\,g_t\,du^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi) + r^2\sin(\psi)^2\,\mathrm{d}(\phi)^2 - g_t\,du^2 + r^2\,\mathrm{d}(\psi)^2$$

```
> b55:=subs(d(psi)^2=0,dr^2=0,du*dr=0,dr=0,d(psi)=0,d(phi)=0,b54);
```

$$b55 := -2\,g_t\,du^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,\mathrm{N}(r)
- 2\,g_t\,du^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi) - g_t\,du^2$$

```
> b56:=simplify(b55);
```

$$b56 := -2\,g_t\,du^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi)\,\mathrm{N}(r)
- 2\,g_t\,du^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P_2}(\psi) - g_t\,du^2$$

```
> b57:=subs(du^2=0,dr=0,d(psi)^2=0,d(phi)=0,b54);
```

$$b57 := 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\frac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du - 2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du$$

```
> b58:=simplify(b57);
```

$$b58 := 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\frac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du - 2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du$$

```
> b59:=subs(du=0,dr^2=0,d(psi)^2=0,d(phi)=0,b54);
```

$$b59 := 2\,\frac{g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M} + 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\frac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr$$

```
> b60:=subs(du^2=0,dr^2=0,d(psi)=0,d(phi)=0,b54);
```

$$b60 := 2\,\frac{g\_t\;du\;dr\;r}{-r+2\,M} + \frac{4\,g\_t\;du\;dr\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)\,r}{-r+2\,M}$$
$$+ \frac{2\,g\_t\;du\;dr\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,r}{-r+2\,M}$$
$$- \frac{4\,g\_t\;du\;dr\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,M}{-r+2\,M}$$
$$+ \frac{4\,g\_t\;du\;dr\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,r}{-r+2\,M}$$

```
> b61:=simplify(b60);
```

$$b61 := 2g\_t\;du\;dr(r + 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)\,r + \varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,\%1\,\mathrm{P\_2}(\psi)\,r$$
$$-\,2\,\varepsilon\,(\frac{\partial}{\partial r}\,\mathrm{f}(r))\,\%1\,\mathrm{P\_2}(\psi)\,M + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,r)/(-r+2\,M)$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$

```
> b62:=subs(du^2=0,dr=0,du=0,d(phi)=0,b54);
```

$$b62 := 2\,r^2\,\varepsilon\,(\frac{\partial}{\partial \psi}\,\alpha(u,\,r,\,\psi))\,\mathrm{d}(\psi)^2 + 2\,r^2\,\mathrm{d}(\psi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r)$$
$$+\,6\,r^2\,\mathrm{d}(\psi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\sin(\psi)^2$$
$$-\,6\,r^2\,\mathrm{d}(\psi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{V}(r)\cos(\psi)^2 + r^2\,\mathrm{d}(\psi)^2$$

```
> b63:=simplify(b62);
```

$$b63 := -r^2\,\mathrm{d}(\psi)^2(-2\,\varepsilon\,(\frac{\partial}{\partial \psi}\,\alpha(u,\,r,\,\psi)) - 2\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r)$$
$$-\,6\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,\mathrm{V}(r) + 12\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,\mathrm{V}(r)\cos(\psi)^2 - 1)$$

```
> b64:=subs(du=0,d(psi)=0,d(psi)^2=0,d(phi)=0,b54);
```

$$b64 := 0$$

```
> b65:=subs(d(psi)=0,dr=0,du=0,b54);
```

$$b65 := -2\,r^2\sin(\psi)^2\,\mathrm{d}(\phi)^2\,\varepsilon\,\alpha\cot(\psi) + 2\,r^2\sin(\psi)^2\,\mathrm{d}(\phi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r)$$
$$-\,6\,r^2\sin(\psi)^3\,\mathrm{d}(\phi)^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\cos(\psi)\cot(\psi)\,\mathrm{V}(r) + r^2\sin(\psi)^2\,\mathrm{d}(\phi)^2$$

```
> b66:=simplify(b65);
```

$$b66 := -r^2\sin(\psi)\,\mathrm{d}(\phi)^2(2\,\varepsilon\,\alpha\cos(\psi) - 2\sin(\psi)\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r)$$
$$+\,6\sin(\psi)\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\cos(\psi)^2\,\mathrm{V}(r) - \sin(\psi))$$

```
> b67:=b56+b58+b59+b61+b63+b66;
```

$$b67 := -2\,g\_t\,du^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)$$
$$-\,2\,g\_t\,du^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi) - g\_t\,du^2$$
$$+\,2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du - 2\,g\_t\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du$$
$$+\,\frac{2\,g\_t\,dr\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M}$$
$$+\,2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr + 2\,g\_t\,du\,dr(r + 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)\,r$$
$$+\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,\%1\,\mathrm{P\_2}(\psi)\,r - 2\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,\%1\,\mathrm{P\_2}(\psi)\,M + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,r)$$
$$/(-r+2\,M) - r^2\,\mathrm{d}(\psi)^2(-2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,r,\,\psi)) - 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) - 6\,\varepsilon\,\%1\,\mathrm{V}(r)$$
$$+\,12\,\varepsilon\,\%1\,\mathrm{V}(r)\cos(\psi)^2 - 1) - r^2\sin(\psi)\,\mathrm{d}(\phi)^2$$
$$(2\,\varepsilon\,\alpha\cos(\psi) - 2\sin(\psi)\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) + 6\sin(\psi)\,\varepsilon\,\%1\cos(\psi)^2\,\mathrm{V}(r) - \sin(\psi))$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$

```
> b68:=subs(g_t=(1-2*M/r),b67);
```

$$b68 := -2\,(1-\frac{2\,M}{r})\,du^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)$$
$$-\,2\,(1-\frac{2\,M}{r})\,du^2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,\mathrm{P\_2}(\psi) - (1-\frac{2\,M}{r})\,du^2$$
$$+\,2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,r,\,\psi))\,du$$
$$-\,2\,(1-\frac{2\,M}{r})\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(r))}\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du$$
$$+\,\frac{2\,(1-\frac{2\,M}{r})\,dr\,\varepsilon\,\mathrm{f}(r)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(r))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M}$$
$$+\,2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\alpha(u,\,r,\,\psi))\,dr + 2\,(1-\frac{2\,M}{r})\,du\,dr(r + 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(r)\,r$$
$$+\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,\%1\,\mathrm{P\_2}(\psi)\,r - 2\,\varepsilon\,(\tfrac{\partial}{\partial r}\,\mathrm{f}(r))\,\%1\,\mathrm{P\_2}(\psi)\,M + 2\,\varepsilon\,\mathrm{f}(r)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,r)$$
$$/(-r+2\,M) - r^2\,\mathrm{d}(\psi)^2(-2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,r,\,\psi)) - 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) - 6\,\varepsilon\,\%1\,\mathrm{V}(r)$$
$$+\,12\,\varepsilon\,\%1\,\mathrm{V}(r)\cos(\psi)^2 - 1) - r^2\sin(\psi)\,\mathrm{d}(\phi)^2$$
$$(2\,\varepsilon\,\alpha\cos(\psi) - 2\sin(\psi)\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(r) + 6\sin(\psi)\,\varepsilon\,\%1\cos(\psi)^2\,\mathrm{V}(r) - \sin(\psi))$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$

```
> b69:=r=R-epsilon*beta(u,R,psi);
```

$$b69 := r = R - \varepsilon\,\beta(u,\,R,\,\psi)$$

```
> b70:=dr=dR-epsilon*diff(beta(u,R,psi),u)*du-epsilon*diff(beta(u,R,psi
> ),R)*dR-epsilon*diff(beta(u,R,psi),psi)*d(psi);
```

$$b70 := dr = dR - \varepsilon\,(\tfrac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,du - \varepsilon\,(\tfrac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR - \varepsilon\,(\tfrac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,\mathrm{d}(\psi)$$

```
> b71:=du*dr=(1-epsilon*diff(beta(u,R,psi),R))*du*dR-epsilon*diff(beta(
> u,R,psi),u)*du^2-epsilon*diff(beta(u,R,psi),psi)*du*d(psi);
```

$$b71 := du\,dr = (1 - \varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi)))\,du\,dR - \varepsilon\,(\frac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,du^2 - \varepsilon\,(\frac{\partial}{\partial \psi}\,\beta(u,\,R,\,\psi))\,du\,\mathrm{d}(\psi)$$

```
> b72:=subs(b71,F(r)=F,f(r)=f,N(r)=N,L(r)=L,V(r)=V,T(r)=T,diff(F(r),r)=
> d_F,diff(f(r),r)=d_f,diff(alpha(u,r,psi),u)=d_alpha(u),diff(alpha(u,r,
> psi),r)=d_alpha(r),diff(alpha(u,r,psi),psi)=d_alpha(psi),b68);
```

$$\begin{aligned}
b72 := &-2\,(1-\frac{2\,M}{r})\,du^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,F)}\,\mathrm{P_2}(\psi)\,N \\
&- 2\,(1-\frac{2\,M}{r})\,du^2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,u+i\,\sigma\,F)}\,\mathrm{P_2}(\psi) - (1-\frac{2\,M}{r})\,du^2 \\
&+ 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d_alpha}(u)\,du - 2\,(1-\frac{2\,M}{r})\,\varepsilon\,f\,e^{(i\,\sigma\,u+i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P_2}(\psi))\,\mathrm{d}(\psi)\,du \\
&+ \frac{2\,(1-\frac{2\,M}{r})\,dr\,\varepsilon\,f\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P_2}(\psi))\,\mathrm{d}(\psi)\,r}{-r+2\,M} + 2\,r^2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d_alpha}(r)\,dr \\
&+ 2\,(1-\frac{2\,M}{r})\,du\,dr(r + 2\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,N\,r \\
&+ \varepsilon\,(\frac{\partial}{\partial r}\,f)\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,r - 2\,\varepsilon\,(\frac{\partial}{\partial r}\,f)\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,M \\
&+ 2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,r)/(-r+2\,M) - r^2\,\mathrm{d}(\psi)^2(-2\,\varepsilon\,\mathrm{d_alpha}(\psi) \\
&- 2\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,T - 6\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,V + 12\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,V\cos(\psi)^2 - 1) \\
&- r^2\sin(\psi)\,\mathrm{d}(\phi)^2(2\,\varepsilon\,\alpha\cos(\psi) - 2\sin(\psi)\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,T \\
&+ 6\sin(\psi)\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\cos(\psi)^2\,V - \sin(\psi))
\end{aligned}$$

```
> b73:=r=R-epsilon*beta(u,R,psi);
```

$$b73 := r = R - \varepsilon\,\beta(u,\,R,\,\psi)$$

```
> b74:=r^2=(R-epsilon*beta(u,R,psi))^2;
```

$$b74 := r^2 = (R - \varepsilon\,\beta(u,\,R,\,\psi))^2$$

```
> b75:=expand(b74);
```

$$b75 := r^2 = R^2 - 2\,\varepsilon\,\beta(u,\,R,\,\psi)\,R + \varepsilon^2\,\beta(u,\,R,\,\psi)^2$$

```
> b76:=subs(epsilon^2=0,b75);
```

$$b76 := r^2 = R^2 - 2\,\varepsilon\,\beta(u,\,R,\,\psi)\,R$$

```
> b77:=subs(b70,b73,b72);
```

$$b77 := -2\,(1 - \frac{2\,M}{R - \%1})\,du^2\,\varepsilon\,e^{(i\,\sigma\,u + i\,\sigma\,F)}\,\mathrm{P\_2}(\psi)\,N$$
$$- 2\,(1 - \frac{2\,M}{R - \%1})\,du^2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,u + i\,\sigma\,F)}\,\mathrm{P\_2}(\psi) - (1 - \frac{2\,M}{R - \%1})\,du^2$$
$$+ 2\,(R - \%1)^2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(u)\,du$$
$$- 2\,(1 - \frac{2\,M}{R - \%1})\,\varepsilon\,f\,e^{(i\,\sigma\,u + i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du$$
$$+ \frac{2\,(1 - \frac{2\,M}{R - \%1})\,\%2\,\varepsilon\,f\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,(R - \%1)}{-R + \%1 + 2\,M}$$
$$+ 2\,(R - \%1)^2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(R - \%1)\,\%2 + 2\,(1 - \frac{2\,M}{R - \%1})\,du\,\%2(R - \%1$$
$$+ 2\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,N\,(R - \%1) + 2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,(R - \%1))/($$
$$-R + \%1 + 2\,M) - (R - \%1)^2\,\mathrm{d}(\psi)^2(-2\,\varepsilon\,\mathrm{d\_alpha}(\psi) - 2\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,T$$
$$- 6\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,V + 12\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,V\cos(\psi)^2 - 1) - (R - \%1)^2\sin(\psi)\,\mathrm{d}(\phi)^2($$
$$2\,\varepsilon\,\alpha\cos(\psi) - 2\sin(\psi)\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,T + 6\sin(\psi)\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\cos(\psi)^2\,V$$
$$- \sin(\psi))$$
$$\%1 := \varepsilon\,\beta(u,\,R,\,\psi)$$
$$\%2 := dR - \varepsilon\,(\frac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,du - \varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR - \varepsilon\,(\frac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,\mathrm{d}(\psi)$$

```
> b78:=subs(epsilon^2=0,epsilon^3=0,epsilon^4=0,epsilon^5=0,simplify(b7
> 7));
```

$$b78 := (-\mathrm{d}(\phi)^2\,R^3\cos(\psi)^2 + \mathrm{d}(\phi)^2\,R^3 - 2\,du^2\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,N\,R$$
$$+ 3\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,\beta(u,\,R,\,\psi)\cos(\psi)^2 - 2\,\varepsilon\,f\,e^{(i\,\sigma\,(u+F))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du\,R$$
$$- 2\,\varepsilon\,f\,e^{(i\,\sigma\,u + i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,dR\,R - 6\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\cos(\psi)^2\,V$$
$$+ 4\,du^2\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,N\,M + 2\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,T$$
$$+ 6\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\cos(\psi)^4\,V - 3\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$+ 4\,\varepsilon\,f\,e^{(i\,\sigma\,(u+F))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du\,M - 2\,du^2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,R$$
$$- 2\,du\,dR\,R - du^2\,R + 2\,du^2\,M + 4\,du^2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,M$$
$$- 3\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,du\,dR\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$- 2\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,T\cos(\psi)^2$$
$$- 4\,du\,dR\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,R + 2\,du\,\varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR\,R$$
$$+ 2\,du\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,\mathrm{d}(\psi)\,R + 6\,\mathrm{d}(\psi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,V$$
$$- 12\,\mathrm{d}(\psi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,V\cos(\psi)^2 + 2\,\mathrm{d}(\psi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,T$$

$+ 2\,\varepsilon\,(\frac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,du^2\,R + 2\,\mathrm{d}(\psi)^2\,R^3\,\varepsilon\,\mathrm{d\_alpha}(\psi)$
$- 2\sin(\psi)\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,\alpha\cos(\psi) + \mathrm{d}(\psi)^2\,R^3 + 2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(u)\,du\,R^3$
$+ 2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(R - \varepsilon\,\beta(u,\,R,\,\psi))\,R^3\,dR - 4\,du\,dR\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,N\,R$
$+ du^2\,\varepsilon\,\beta(u,\,R,\,\psi))/(R - \varepsilon\,\beta(u,\,R,\,\psi))$

```
> b79:=subs((R-epsilon*beta(u,R,psi))^(-1)=1/R+epsilon*beta(u,R,psi)/R^
> 2,b78);
```

$$\begin{aligned}
b79 := (&-\mathrm{d}(\phi)^2\,R^3\cos(\psi)^2 + \mathrm{d}(\phi)^2\,R^3 - 2\,du^2\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,N\,R\\
&+ 3\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,\beta(u,\,R,\,\psi)\cos(\psi)^2 - 2\,\varepsilon\,f\,e^{(i\,\sigma\,(u+F))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P_2}(\psi))\,\mathrm{d}(\psi)\,du\,R\\
&- 2\,\varepsilon\,f\,e^{(i\,\sigma\,u+i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P_2}(\psi))\,\mathrm{d}(\psi)\,dR\,R - 6\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\cos(\psi)^2\,V\\
&+ 4\,du^2\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,N\,M + 2\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,T\\
&+ 6\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\cos(\psi)^4\,V - 3\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,\beta(u,\,R,\,\psi)\\
&+ 4\,\varepsilon\,f\,e^{(i\,\sigma\,(u+F))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P_2}(\psi))\,\mathrm{d}(\psi)\,du\,M - 2\,du^2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,R\\
&- 2\,du\,dR\,R - du^2\,R + 2\,du^2\,M + 4\,du^2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,M\\
&- 3\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,du\,dR\,\varepsilon\,\beta(u,\,R,\,\psi)\\
&- 2\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,T\cos(\psi)^2\\
&- 4\,du\,dR\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,R + 2\,du\,\varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR\,R\\
&+ 2\,du\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,\mathrm{d}(\psi)\,R + 6\,\mathrm{d}(\psi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,V\\
&- 12\,\mathrm{d}(\psi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,V\cos(\psi)^2 + 2\,\mathrm{d}(\psi)^2\,R^3\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P_2}(\psi)\,T
\end{aligned}$$

$+ 2\,\varepsilon\,(\frac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,du^2\,R + 2\,\mathrm{d}(\psi)^2\,R^3\,\varepsilon\,\mathrm{d\_alpha}(\psi)$
$- 2\sin(\psi)\,\mathrm{d}(\phi)^2\,R^3\,\varepsilon\,\alpha\cos(\psi) + \mathrm{d}(\psi)^2\,R^3 + 2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(u)\,du\,R^3$
$+ 2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(R - \varepsilon\,\beta(u,\,R,\,\psi))\,R^3\,dR - 4\,du\,dR\,\varepsilon\,e^{(i\,\sigma\,(u+F))}\,\mathrm{P\_2}(\psi)\,N\,R$
$+ du^2\,\varepsilon\,\beta(u,\,R,\,\psi))(\frac{1}{R} + \frac{\varepsilon\,\beta(u,\,R,\,\psi)}{R^2})$

```
> b80:=subs(epsilon^2=0,epsilon^3=0,expand(b79));
```

$$\begin{aligned}
b80 := &-\mathrm{d}(\phi)^2\,R^2\cos(\psi)^2 - 2\,du\,dR + 2\,\mathrm{d}(\phi)^2\,R\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi)\\
&+ \frac{2\,du^2\,M\,\varepsilon\,\beta(u,\,R,\,\psi)}{R^2} + \frac{2\,du^2\,M}{R} + \frac{4\,du^2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P_2}(\psi)\,M}{R}\\
&+ 2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,du\,\mathrm{d}(\psi) - 6\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\cos(\psi)^2\,V\\
&+ 2\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P_2}(\psi)\,T + 6\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\cos(\psi)^4\,V\\
&- 2\,du^2\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P_2}(\psi) - 4\,du\,dR\,\varepsilon\,f\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P_2}(\psi)
\end{aligned}$$

$$- 12\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,V\cos(\psi)^2 + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P\_2}(\psi)\,T$$
$$- 2\,du^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P\_2}(\psi)\,N + \frac{4\,\varepsilon\,f\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du\,M}{R}$$
$$+ 2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(u)\,du\,R^2 + 2\,du\,\varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR$$
$$+ 6\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,V + 2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(R - \varepsilon\,\beta(u,\,R,\,\psi))\,R^2\,dR$$
$$- 2\,\varepsilon\,f\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du + \frac{4\,du^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P\_2}(\psi)\,N\,M}{R}$$
$$- 2\,\varepsilon\,f\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,dR - 2\sin(\psi)\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,\alpha\cos(\psi)$$
$$- 2\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P\_2}(\psi)\,T\cos(\psi)^2 - 4\,du\,dR\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,F)}\,\mathrm{P\_2}(\psi)\,N$$
$$+ 2\,\varepsilon\,(\frac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,du^2 - 2\,\mathrm{d}(\psi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) - 2\,\mathrm{d}(\phi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$+ 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,\mathrm{d\_alpha}(\psi) + \mathrm{d}(\phi)^2\,R^2 + \mathrm{d}(\psi)^2\,R^2 - du^2$$

```
> b81:=subs(F=F(R),f=f(R),N=N(R),L=L(R),V=V(R),T=T(R),d_F=diff(F(R),R),
> d_f=diff(f(R),R),d_alpha(u)=diff(alpha(u,R,psi),u),d_alpha(r)=diff(u,R
> ,psi),d_alpha(psi)=diff(alpha(u,R,psi),psi),(R-epsilon*beta(u,R,psi))^
> (-1)=R+epsilon*beta(u,R,psi),b80);
```

$$b81 := -\mathrm{d}(\phi)^2\,R^2\cos(\psi)^2 + 6\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\cos(\psi)^4\,\mathrm{V}(R) - 2\,du\,dR$$
$$+ 2\,\mathrm{d}(\phi)^2\,R\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi) + \frac{2\,du^2\,M\,\varepsilon\,\beta(u,\,R,\,\psi)}{R^2} + \frac{2\,du^2\,M}{R}$$
$$+ 2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,du\,\mathrm{d}(\psi) + 2\,du\,\varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR$$
$$- 4\,du\,dR\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R) - 2\,du^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)$$
$$- 4\,du\,dR\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)$$
$$+ \frac{4\,du^2\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,M}{R}$$
$$+ \frac{4\,\varepsilon\,\mathrm{f}(R)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du\,M}{R}$$
$$+ 6\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R)$$
$$- 2\,\varepsilon\,\mathrm{f}(R)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du$$
$$- 2\,\varepsilon\,\mathrm{f}(R)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,dR$$
$$+ 2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(R - \varepsilon\,\beta(u,\,R,\,\psi))\,R^2\,dR$$
$$+ 2\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$- 6\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\cos(\psi)^2\,\mathrm{V}(R)$$
$$+ 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$

$$- 12\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R)\cos(\psi)^2 - 2\sin(\psi)\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,\alpha\cos(\psi)$$
$$+ 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi)) + 2\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,du^2 - 2\,\mathrm{d}(\psi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$- 2\,\mathrm{d}(\phi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) - 2\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2$$
$$+ \frac{4\,du^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,M}{R}$$
$$- 2\,du^2\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi) + \mathrm{d}(\phi)^2\,R^2$$
$$+ 2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,R,\,\psi))\,du\,R^2 + \mathrm{d}(\psi)^2\,R^2 - du^2$$

```
> b82:=subs(d(psi)^2=0,dR^2=0,du*dR=0,dR=0,d(psi)=0,d(phi)=0,b81);
```

$$b82 := 2\,\frac{du^2\,M\,\varepsilon\,\beta(u,\,R,\,\psi)}{R^2} + \frac{2\,du^2\,M}{R} - 2\,du^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)$$
$$+ \frac{4\,du^2\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,M}{R} + 2\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,du^2$$
$$+ \frac{4\,du^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,M}{R}$$
$$- 2\,du^2\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi) - du^2$$

```
> b83:=simplify(b82);
```

$$b83 := -du^2(-2\,M\,\varepsilon\,\beta(u,\,R,\,\psi) - 2\,M\,R + 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,R^2$$
$$- 4\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,M\,R - 2\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,R^2 - 4\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,M\,R$$
$$+ 2\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,R^2 + R^2)/R^2$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> b84:=subs(du^2=0,dR=0,d(psi)^2=0,d(phi)=0,b81);
```

$$b84 := 2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,du\,\mathrm{d}(\psi) + \frac{4\,\varepsilon\,\mathrm{f}(R)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du\,M}{R}$$
$$- 2\,\varepsilon\,\mathrm{f}(R)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,du + 2\,\mathrm{d}(\psi)\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,R,\,\psi))\,du\,R^2$$

```
> b85:=simplify(b84);
```

$$b85 := 2du\,\varepsilon\,\mathrm{d}(\psi)((\tfrac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,R + 2\,\mathrm{f}(R)\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,M$$
$$- \mathrm{f}(R)\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,R + (\tfrac{\partial}{\partial u}\,\alpha(u,\,R,\,\psi))\,R^3)/R$$

```
> b86:=subs(du=0,dR^2=0,d(psi)^2=0,d(phi)=0,b81);
```

$$b86 := -2\,\varepsilon\,\mathrm{f}(R)\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,\mathrm{d}(\psi)\,dR$$
$$+ 2\,\mathrm{d}(\psi)\,\varepsilon\,\mathrm{d\_alpha}(R - \varepsilon\,\beta(u,\,R,\,\psi))\,R^2\,dR$$

```
> b87:=subs(du^2=0,dR^2=0,d(psi)=0,d(phi)=0,b81);
```

$$b87 := -2\,du\,dR + 2\,du\,\varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR - 4\,du\,dR\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)$$
$$-\,4\,du\,dR\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)$$

```
> b88:=simplify(b87);
```

$$b88 := -2\,du\,dR + 2\,du\,\varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR - 4\,du\,dR\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)$$
$$-\,4\,du\,dR\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)$$

```
> b89:=subs(du^2=0,dR=0,du=0,d(phi)=0,b81);
```

$$b89 := 6\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R) + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$-\,12\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R)\cos(\psi)^2 + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi))$$
$$-\,2\,\mathrm{d}(\psi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) + \mathrm{d}(\psi)^2\,R^2$$

```
> b90:=simplify(b89);
```

$$b90 := 6\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R) + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$-\,12\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R)\cos(\psi)^2 + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi))$$
$$-\,2\,\mathrm{d}(\psi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) + \mathrm{d}(\psi)^2\,R^2$$

```
> b91:=subs(d(psi)=0,dR=0,du=0,b81);
```

$$b91 := -\mathrm{d}(\phi)^2\,R^2\cos(\psi)^2 + 6\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\cos(\psi)^4\,\mathrm{V}(R)$$
$$+\,2\,\mathrm{d}(\phi)^2\,R\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$-\,6\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\cos(\psi)^2\,\mathrm{V}(R) - 2\sin(\psi)\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,\alpha\cos(\psi)$$
$$-\,2\,\mathrm{d}(\phi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) - 2\,\mathrm{d}(\phi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u)}\,e^{(i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2$$
$$+\,\mathrm{d}(\phi)^2\,R^2$$

```
> b92:=simplify(b91);
```

$$b92 := -\mathrm{d}(\phi)^2\,R(R\cos(\psi)^2 - 6\,R\,\varepsilon\,\%1\cos(\psi)^4\,\mathrm{V}(R) - 2\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$-\,2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) + 6\,R\,\varepsilon\,\%1\,\mathrm{V}(R)\cos(\psi)^2 + 2\sin(\psi)\,R\,\varepsilon\,\alpha\cos(\psi)$$
$$+\,2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2 - R)$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> b93:=ds^2=b83+b85+b88+b90+b92;
```

$$b93 := ds^2 = -du^2(-2\,M\,\varepsilon\,\beta(u,\,R,\,\psi) - 2\,M\,R + 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,R^2$$
$$-\,4\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,M\,R - 2\,\varepsilon\,(\frac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,R^2 - 4\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,M\,R$$
$$+\,2\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,R^2 + R^2)/R^2 + 2\,du\,\varepsilon\,\mathrm{d}(\psi)((\frac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi))\,R$$
$$+\,2\,\mathrm{f}(R)\,\%1\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,M - \mathrm{f}(R)\,\%1\,(\frac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,R + (\frac{\partial}{\partial u}\,\alpha(u,\,R,\,\psi))\,R^3)/R$$

$- 2\,du\,dR + 2\,du\,\varepsilon\,(\frac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR - 4\,du\,dR\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)$
$- 4\,du\,dR\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%2\,\mathrm{P\_2}(\psi) + 6\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,\%2\,\mathrm{V}(R)$
$+ 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,\%2\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) - 12\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,\%2\,\mathrm{V}(R)\cos(\psi)^2$
$+ 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi)) - 2\,\mathrm{d}(\psi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) + \mathrm{d}(\psi)^2\,R^2 - \mathrm{d}(\phi)^2\,R($
$R\cos(\psi)^2 - 6\,R\,\varepsilon\,\%1\cos(\psi)^4\,\mathrm{V}(R) - 2\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi)$
$- 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) + 6\,R\,\varepsilon\,\%1\,\mathrm{V}(R)\cos(\psi)^2 + 2\sin(\psi)\,R\,\varepsilon\,\alpha\cos(\psi)$
$+ 2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2 - R)$
$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$
$\%2 := e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}$

```
> b94:=subs(du^2=0,dR=0,du=0,d(phi)=0,b93);
```

$$b94 := ds^2 = 6\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R) + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$- 12\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R)\cos(\psi)^2 + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi))$$
$$- 2\,\mathrm{d}(\psi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) + \mathrm{d}(\psi)^2\,R^2$$

```
> b95:=coeff(rhs(b94),d(psi)^2);
```

$$b95 := 6\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R) + 2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$- 12\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R)\cos(\psi)^2 + 2\,R^2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi)) - 2\,\varepsilon\,\beta(u,\,R,\,\psi)\,R$$
$$+ R^2$$

```
> b96:=subs(d(psi)=0,dr=0,du=0,b93);
```

$$b96 := ds^2 = -\mathrm{d}(\phi)^2\,R(R\cos(\psi)^2 - 6\,R\,\varepsilon\,\%1\cos(\psi)^4\,\mathrm{V}(R) - 2\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$- 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) + 6\,R\,\varepsilon\,\%1\,\mathrm{V}(R)\cos(\psi)^2 + 2\sin(\psi)\,R\,\varepsilon\,\alpha\cos(\psi)$$
$$+ 2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2 - R)$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> b97:=coeff(rhs(b96),d(phi)^2);
```

$$b97 := -R(R\cos(\psi)^2 - 6\,R\,\varepsilon\,\%1\cos(\psi)^4\,\mathrm{V}(R) - 2\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$- 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) + 6\,R\,\varepsilon\,\%1\,\mathrm{V}(R)\cos(\psi)^2 + 2\sin(\psi)\,R\,\varepsilon\,\alpha\cos(\psi)$$
$$+ 2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2 - R)$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> b98:=b95*b97=(R^4+4*R^3*epsilon*beta(u,R,psi))*sin(psi)^2;
```

$$b98 := -(6\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R) + 2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$- 12\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R)\cos(\psi)^2 + 2\,R^2\,\varepsilon\,(\frac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi)) - 2\,\varepsilon\,\beta(u,\,R,\,\psi)\,R$$
$$+ R^2)R(R\cos(\psi)^2 - 6\,R\,\varepsilon\,\%1\cos(\psi)^4\,\mathrm{V}(R) - 2\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$- 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) + 6\,R\,\varepsilon\,\%1\,\mathrm{V}(R)\cos(\psi)^2 + 2\sin(\psi)\,R\,\varepsilon\,\alpha\cos(\psi)$$

$$+ 2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2 - R) =$$
$$(R^4 + 4\,R^3\,\varepsilon\,\beta(u,\,R,\,\psi))\sin(\psi)^2$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> b99:=subs(epsilon^2=0,epsilon^3=0,simplify(b98));
```

$$\mathit{b99} := -R^2(-R^2 - 6\,R^2\,\varepsilon\,\%1\,\mathrm{V}(R) + R^2\cos(\psi)^2 + 2\,R^2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi))\cos(\psi)^2$$
$$- 4\,R\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi) + 4\,R^2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2$$
$$+ 24\,R^2\,\varepsilon\,\%1\,\mathrm{V}(R)\cos(\psi)^2 - 4\,R^2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) + 2\sin(\psi)\,R^2\,\varepsilon\,\alpha\cos(\psi)$$
$$- 18\,R^2\,\varepsilon\,\%1\cos(\psi)^4\,\mathrm{V}(R) - 2\,R^2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi)) + 4\,\varepsilon\,\beta(u,\,R,\,\psi)\,R) =$$
$$-R^3\,(R + 4\,\varepsilon\,\beta(u,\,R,\,\psi))\,(-1 + \cos(\psi)^2)$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> b100:=solve(b99,beta(u,R,psi));
```

$$\mathit{b100} := -\frac{1}{4}R(3\,\%1\,\mathrm{V}(R) - (\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi))\cos(\psi)^2 - 2\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2$$
$$- 12\,\%1\,\mathrm{V}(R)\cos(\psi)^2 + 2\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) - \sin(\psi)\,\alpha\cos(\psi)$$
$$+ 9\,\%1\cos(\psi)^4\,\mathrm{V}(R) + (\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi)))\Big/(-1 + \cos(\psi)^2)$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> simplify(coeff(rhs(b93),epsilon,0));
```

$$-\frac{-2\,du^2\,M + du^2\,R + 2\,du\,dR\,R - \mathrm{d}(\psi)^2\,R^3 + \mathrm{d}(\phi)^2\,R^3\cos(\psi)^2 - \mathrm{d}(\phi)^2\,R^3}{R}$$

```
> simplify(coeff(rhs(b93),epsilon,1));
```

$$2(2\,du\,\mathrm{d}(\psi)\,R\,\mathrm{f}(R)\,\%1\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))\,M - du\,\mathrm{d}(\psi)\,R^2\,\mathrm{f}(R)\,\%1\,(\tfrac{\partial}{\partial\psi}\,\mathrm{P\_2}(\psi))$$
$$+ 3\,\mathrm{d}(\phi)^2\,R^4\,\%1\cos(\psi)^4\,\mathrm{V}(R) - \mathrm{d}(\phi)^2\,R^4\sin(\psi)\,\alpha\cos(\psi)$$
$$+ \mathrm{d}(\phi)^2\,R^4\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) - 3\,\mathrm{d}(\phi)^2\,R^4\,\%1\,\mathrm{V}(R)\cos(\psi)^2$$
$$- \mathrm{d}(\phi)^2\,R^4\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2 + du\,\mathrm{d}(\psi)\,R^4\,(\tfrac{\partial}{\partial u}\,\alpha(u,\,R,\,\psi))$$
$$+ \mathrm{d}(\phi)^2\,R^3\cos(\psi)^2\,\beta(u,\,R,\,\psi) - \mathrm{d}(\phi)^2\,R^3\,\beta(u,\,R,\,\psi) + \mathrm{d}(\psi)^2\,R^4\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi))$$
$$- \mathrm{d}(\psi)^2\,R^3\,\beta(u,\,R,\,\psi) + du^2\,M\,\beta(u,\,R,\,\psi) + du^2\,(\tfrac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,R^2$$
$$+ du\,\mathrm{d}(\psi)\,R^2\,(\tfrac{\partial}{\partial\psi}\,\beta(u,\,R,\,\psi)) - du^2\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,R^2$$
$$+ du\,(\tfrac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR\,R^2 - du^2\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,R^2$$
$$+ 2\,du^2\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,M\,R + 2\,du^2\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,M\,R$$
$$- 2\,du\,dR\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,R^2 - 2\,du\,dR\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,R^2$$

$+ \mathrm{d}(\psi)^2 R^4 \%1\, \mathrm{P\_2}(\psi)\, \mathrm{T}(R) - 6\, \mathrm{d}(\psi)^2 R^4 \%1\, \mathrm{V}(R) \cos(\psi)^2 + 3\, \mathrm{d}(\psi)^2 R^4 \%1\, \mathrm{V}(R))$
$/R^2$
$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$

```
> b101:=subs(d(psi)^2=0,dR^2=0,du*dR=0,dR=0,d(psi)=0,d(phi)=0,b93);
```

$$b101 := ds^2 = -du^2(-2\,M\,\varepsilon\,\beta(u,\,R,\,\psi) - 2\,M\,R + 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,R^2 - 4\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,M\,R - 2\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,R^2 - 4\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,M\,R + 2\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,R^2 + R^2)/R^2$$
$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$

```
> b102:=simplify(b101);
```

$$b102 := ds^2 = -du^2(-2\,M\,\varepsilon\,\beta(u,\,R,\,\psi) - 2\,M\,R + 2\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,R^2 - 4\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,M\,R - 2\,\varepsilon\,(\tfrac{\partial}{\partial u}\,\beta(u,\,R,\,\psi))\,R^2 - 4\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R)\,M\,R + 2\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,\%1\,\mathrm{P\_2}(\psi)\,R^2 + R^2)/R^2$$
$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$

```
> b103:=subs(du^2=0,dR=0,d(psi)^2=0,d(phi)=0,b93);
```

$$b103 := ds^2 = 2du\,\varepsilon\,\mathrm{d}(\psi)((\tfrac{\partial}{\partial \psi}\,\beta(u,\,R,\,\psi))\,R + 2\,\mathrm{f}(R)\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,(\tfrac{\partial}{\partial \psi}\,\mathrm{P\_2}(\psi))\,M - \mathrm{f}(R)\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,(\tfrac{\partial}{\partial \psi}\,\mathrm{P\_2}(\psi))\,R + (\tfrac{\partial}{\partial u}\,\alpha(u,\,R,\,\psi))\,R^3)/R$$

```
> b104:=simplify(b103);
```

$$b104 := ds^2 = 2du\,\varepsilon\,\mathrm{d}(\psi)((\tfrac{\partial}{\partial \psi}\,\beta(u,\,R,\,\psi))\,R + 2\,\mathrm{f}(R)\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,(\tfrac{\partial}{\partial \psi}\,\mathrm{P\_2}(\psi))\,M - \mathrm{f}(R)\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,(\tfrac{\partial}{\partial \psi}\,\mathrm{P\_2}(\psi))\,R + (\tfrac{\partial}{\partial u}\,\alpha(u,\,R,\,\psi))\,R^3)/R$$

```
> b105:=subs(du=0,dR^2=0,d(psi)^2=0,d(phi)=0,b93);
```

$$b105 := ds^2 = 0$$

```
> b106:=subs(du^2=0,dR^2=0,d(psi)=0,d(phi)=0,b93);
```

$$b106 := ds^2 = -2\,du\,dR + 2\,du\,\varepsilon\,(\tfrac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR - 4\,du\,dR\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R) - 4\,du\,dR\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)$$

```
> b107:=simplify(b106);
```

$$b107 := ds^2 = -2\,du\,dR + 2\,du\,\varepsilon\,(\tfrac{\partial}{\partial R}\,\beta(u,\,R,\,\psi))\,dR - 4\,du\,dR\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,\mathrm{P\_2}(\psi)\,\mathrm{N}(R) - 4\,du\,dR\,\varepsilon\,\mathrm{f}(R)\,i\,\sigma\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,\mathrm{P\_2}(\psi)$$

```
> b108:=subs(du^2=0,dR=0,du=0,d(phi)=0,b93);
```

$$b108 := ds^2 = 6\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R) + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$-\,12\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,u+i\,\sigma\,\mathrm{F}(R))}\,\mathrm{V}(R)\cos(\psi)^2 + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi))$$
$$-\,2\,\mathrm{d}(\psi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) + \mathrm{d}(\psi)^2\,R^2$$

```
> b109:=simplify(b108);
```

$$b109 := ds^2 = 6\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,\mathrm{V}(R) + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$-\,12\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,\mathrm{V}(R)\cos(\psi)^2 + 2\,\mathrm{d}(\psi)^2\,R^2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi))$$
$$-\,2\,\mathrm{d}(\psi)^2\,R\,\varepsilon\,\beta(u,\,R,\,\psi) + \mathrm{d}(\psi)^2\,R^2$$

```
> b110:=subs(d(psi)=0,dR=0,du=0,b93);
```

$$b110 := ds^2 = -\mathrm{d}(\phi)^2\,R(R\cos(\psi)^2 - 6\,R\,\varepsilon\,\%1\cos(\psi)^4\,\mathrm{V}(R) - 2\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$-\,2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) + 6\,R\,\varepsilon\,\%1\,\mathrm{V}(R)\cos(\psi)^2 + 2\sin(\psi)\,R\,\varepsilon\,\alpha\cos(\psi)$$
$$+\,2\,\varepsilon\,\beta(u,\,R,\,\psi) + 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2 - R)$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> b111:=simplify(coeff(rhs(b109),d(psi)^2));
```

$$b111 := 6\,R^2\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,\mathrm{V}(R) + 2\,R^2\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)$$
$$-\,12\,R^2\,\varepsilon\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}\,\mathrm{V}(R)\cos(\psi)^2 + 2\,R^2\,\varepsilon\,(\tfrac{\partial}{\partial\psi}\,\alpha(u,\,R,\,\psi)) - 2\,\varepsilon\,\beta(u,\,R,\,\psi)\,R$$
$$+\,R^2$$

```
> b112:=simplify(coeff(rhs(b110),d(phi)^2));
```

$$b112 := R(-R\cos(\psi)^2 + 6\,R\,\varepsilon\,\%1\cos(\psi)^4\,\mathrm{V}(R) + 2\cos(\psi)^2\,\varepsilon\,\beta(u,\,R,\,\psi)$$
$$+\,2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R) - 6\,R\,\varepsilon\,\%1\,\mathrm{V}(R)\cos(\psi)^2 - 2\sin(\psi)\,R\,\varepsilon\,\alpha\cos(\psi)$$
$$-\,2\,\varepsilon\,\beta(u,\,R,\,\psi) - 2\,R\,\varepsilon\,\%1\,\mathrm{P\_2}(\psi)\,\mathrm{T}(R)\cos(\psi)^2 + R)$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

Appendix D

Maple computer program 4

This program transforms odd-parity metric perturbations of a Schwarzschild black hole to Bondi-Sachs form. See chapter 5 sec. 5.5

```
> a1:=-(1-2*M/r)*dt^2+(1-2*M/r)^(-1)*dr^2+r^2*d(theta)^2+r^2*sin(theta)
> ^2*(d(phi)-omega*dt-q_(2)*dr-q_(3)*d(theta))^2;
```

$$a1 := -(1-\frac{2\,M}{r})\,dt^2 + \frac{dr^2}{1-\frac{2\,M}{r}} + r^2\,\mathrm{d}(\theta)^2 + r^2\sin(\theta)^2\,(\mathrm{d}(\phi) - \omega\,dt - \mathrm{q\_}(2)\,dr - \mathrm{q\_}(3)\,\mathrm{d}(\theta))^2$$

```
> a2:=u=t-F(r);
```

$$a2 := u = t - \mathrm{F}(r)$$

```
> a3:=dt=du+diff(F(r),r)*dr;
```

$$a3 := dt = du + (\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr$$

```
> a4:=subs(a3,a1);
```

$$a4 := -(1-\frac{2\,M}{r})\,(du + (\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr)^2 + \frac{dr^2}{1-\frac{2\,M}{r}} + r^2\,\mathrm{d}(\theta)^2 + r^2\sin(\theta)^2\,(\mathrm{d}(\phi) - \omega\,(du + (\tfrac{\partial}{\partial r}\,\mathrm{F}(r))\,dr) - \mathrm{q\_}(2)\,dr - \mathrm{q\_}(3)\,\mathrm{d}(\theta))^2$$

```
> a5:=-(1-2*M/r)*du^2-2*(1-2*M/r)*diff(F(r),r)*du*dr+r^2*d(theta)^2+((1
> -2*M/r)^(-1)-(1-2*M/r)*diff(F(r),r)^2)*dr^2+r^2*sin(theta)^2*(d(phi)-o
> mega(r)*du-(omega(r)*diff(F(r),r)+q_2(r))*dr-q_3(r)*d(theta))^2;
```

$$a5 := -(1 - \frac{2\,M}{r})\,du^2 - 2\,(1 - \frac{2\,M}{r})\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr + r^2\,\mathrm{d}(\theta)^2$$
$$+ \left(\frac{1}{1 - \frac{2\,M}{r}} - (1 - \frac{2\,M}{r})\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))^2\right)\,dr^2$$
$$+ r^2\sin(\theta)^2\,(\mathrm{d}(\phi) - \omega(r)\,du - (\omega(r)\,(\frac{\partial}{\partial r}\,\mathrm{F}(r)) + \mathrm{q\_2}(r))\,dr - \mathrm{q\_3}(r)\,\mathrm{d}(\theta))^2$$

```
> a6:=((1-2*M/r)^(-1)-(1-2*M/r)*diff(F(r),r)^2)=0;
```

$$a6 := \frac{1}{1 - \frac{2\,M}{r}} - (1 - \frac{2\,M}{r})\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))^2 = 0$$

```
> a7:=dsolve(a6);
```

$$a7 := \mathrm{F}(r) = r + 2\,M\ln(r - 2\,M) + \_C1,\ \mathrm{F}(r) = -r - 2\,M\ln(r - 2\,M) + \_C1$$

```
> a8:=subs(((1-2*M/r)^(-1)-(1-2*M/r)*diff(F(r),r)^2)=0,omega(r)=epsilon
> *omega(r)*(3*cos(theta))*E,q_2(r)=epsilon*q_2(r)*3*(cos(theta))*E,q_3(
> r)=epsilon*q_3(r)*3*(sin(theta))*E,a5);
```

$$a8 := -(1 - \frac{2\,M}{r})\,du^2 - 2\,(1 - \frac{2\,M}{r})\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr + r^2\,\mathrm{d}(\theta)^2 + r^2\sin(\theta)^2(\mathrm{d}(\phi)$$
$$- 3\,\varepsilon\,\omega(r)\cos(\theta)\,E\,du - (3\,\varepsilon\,\omega(r)\cos(\theta)\,E\,(\frac{\partial}{\partial r}\,\mathrm{F}(r)) + 3\,\varepsilon\,\mathrm{q\_2}(r)\cos(\theta)\,E)\,dr$$
$$- 3\,\varepsilon\,\mathrm{q\_3}(r)\sin(\theta)\,E\,\mathrm{d}(\theta))^2$$

```
> a9:=psi=phi+g(r,theta,u);
```

$$a9 := \psi = \phi + \mathrm{g}(r,\,\theta,\,u)$$

```
> a10:=d(phi)=d(psi)-diff(g(r,theta,u),r)*dr-diff(g(r,theta,u),theta)*d
> (theta)-diff(g(r,theta,u),u)*du;
```

$$a10 := \mathrm{d}(\phi) = \mathrm{d}(\psi) - (\frac{\partial}{\partial r}\,\mathrm{g}(r,\,\theta,\,u))\,dr - (\frac{\partial}{\partial \theta}\,\mathrm{g}(r,\,\theta,\,u))\,\mathrm{d}(\theta) - (\frac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,du$$

```
> a11:=subs(a10,a8);
```

$$a11 := -(1 - \frac{2\,M}{r})\,du^2 - 2\,(1 - \frac{2\,M}{r})\,(\frac{\partial}{\partial r}\,\mathrm{F}(r))\,du\,dr + r^2\,\mathrm{d}(\theta)^2 + r^2\sin(\theta)^2(\mathrm{d}(\psi)$$
$$- (\frac{\partial}{\partial r}\,\mathrm{g}(r,\,\theta,\,u))\,dr - (\frac{\partial}{\partial \theta}\,\mathrm{g}(r,\,\theta,\,u))\,\mathrm{d}(\theta) - (\frac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,du - 3\,\varepsilon\,\omega(r)\cos(\theta)\,E\,du$$
$$- (3\,\varepsilon\,\omega(r)\cos(\theta)\,E\,(\frac{\partial}{\partial r}\,\mathrm{F}(r)) + 3\,\varepsilon\,\mathrm{q\_2}(r)\cos(\theta)\,E)\,dr - 3\,\varepsilon\,\mathrm{q\_3}(r)\sin(\theta)\,E\,\mathrm{d}(\theta))$$
^{2}

```
> a12:=subs(F(r)=r+2*M*ln(r-2*M),a11);
```

$$a12 := -(1-\frac{2\,M}{r})\,du^2 - 2\,(1-\frac{2\,M}{r})\,(\frac{\partial}{\partial r}\,(r+2\,M\ln(r-2\,M)))\,du\,dr + r^2\,\mathrm{d}(\theta)^2 + r^2$$
$$\sin(\theta)^2(\mathrm{d}(\psi) - (\frac{\partial}{\partial r}\,\mathrm{g}(r,\,\theta,\,u))\,dr - (\frac{\partial}{\partial \theta}\,\mathrm{g}(r,\,\theta,\,u))\,\mathrm{d}(\theta) - (\frac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,du$$
$$-\,3\,\varepsilon\,\omega(r)\cos(\theta)\,E\,du$$
$$-\,(3\,\varepsilon\,\omega(r)\cos(\theta)\,E\,(\frac{\partial}{\partial r}\,(r+2\,M\ln(r-2\,M))) + 3\,\varepsilon\,\mathrm{q\_2}(r)\cos(\theta)\,E)\,dr$$
$$-\,3\,\varepsilon\,\mathrm{q\_3}(r)\sin(\theta)\,E\,\mathrm{d}(\theta))^2$$

```
> a13:=diff(g(r,theta,u),r)+epsilon*omega(r)*(3*cos(theta))*E*(1+2*M/(r
> -2*M))+3*epsilon*q_2(r)*(cos(theta))*E=0;
```

$$a13 := (\frac{\partial}{\partial r}\,\mathrm{g}(r,\,\theta,\,u)) + 3\,\varepsilon\,\omega(r)\cos(\theta)\,E\,(1+\frac{2\,M}{r-2\,M}) + 3\,\varepsilon\,\mathrm{q\_2}(r)\cos(\theta)\,E = 0$$

```
> a14:=dsolve(a13,g(r,theta,u));
```

$$a14 := \mathrm{g}(r,\,\theta,\,u) = \int 3\,\frac{\varepsilon\cos(\theta)\,E\,(\omega(r)\,r + \mathrm{q\_2}(r)\,r - 2\,\mathrm{q\_2}(r)\,M)}{-r+2\,M}\,dr + \mathrm{\_F1}(\theta,\,u)$$

```
> a15:=factor(subs(E=e^(i*sigma*(u+F(r))),a14));
```

$$a15 := \mathrm{g}(r,\,\theta,\,u) =$$
$$\int 3\,\frac{\varepsilon\cos(\theta)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,(\omega(r)\,r + \mathrm{q\_2}(r)\,r - 2\,\mathrm{q\_2}(r)\,M)}{-r+2\,M}\,dr + \mathrm{\_F1}(\theta,\,u)$$

```
> a16:=g(r,theta,u)=-3*epsilon*cos(theta)*(int(((e^(i*sigma*(u+F(r))))*
> (-omega(r)*r-q_2(r)*r+2*q_2(r)*M))/(2*M-r),r));
```

$$a16 := \mathrm{g}(r,\,\theta,\,u) = -3\,\varepsilon\cos(\theta)\int \frac{e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,(-\omega(r)\,r - \mathrm{q\_2}(r)\,r + 2\,\mathrm{q\_2}(r)\,M)}{-r+2\,M}\,dr$$

```
> a17:=subs(diff(g(r,theta,u),r)=0,3*epsilon*q_2(r)*cos(theta)*E=0,(-3*
> epsilon*omega(r)*cos(theta)*E*(1+2*M/(r-2*M)))*dr=0,E=e^(i*sigma*(u+F(
> r))),a12);
```

$$a17 := -(1-\frac{2\,M}{r})\,du^2 - 2\,(1-\frac{2\,M}{r})\,(1+\frac{2\,M}{r-2\,M})\,du\,dr + r^2\,\mathrm{d}(\theta)^2 + r^2\sin(\theta)^2(\mathrm{d}(\psi)$$
$$-\,(\frac{\partial}{\partial \theta}\,\mathrm{g}(r,\,\theta,\,u))\,\mathrm{d}(\theta) - (\frac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,du - 3\,\varepsilon\,\omega(r)\cos(\theta)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,du$$
$$-\,3\,\varepsilon\,\mathrm{q\_3}(r)\sin(\theta)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,\mathrm{d}(\theta))^2$$

```
> a18:=simplify(a17);
```

$$a18 := (-2\,dr\,du\,r + 9\,r^3\,\varepsilon^2\,\mathrm{q\_3}(r)^2\,\%1\,\mathrm{d}(\theta)^2\cos(\theta)^4 - du^2\,r + 2\,du^2\,M + r^3\,\mathrm{d}(\theta)^2$$
$$-\,2\,r^3\,\mathrm{d}(\psi)\,\%2\,\mathrm{d}(\theta) - 2\,r^3\,\mathrm{d}(\psi)\,(\frac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,du - r^3\,\%2^2\,\mathrm{d}(\theta)^2\cos(\theta)^2$$
$$-\,r^3\,(\frac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))^2\,du^2\cos(\theta)^2 + r^3\,\mathrm{d}(\psi)^2$$
$$-\,6\,r^3\sin(\theta)\,(\frac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,du\,\varepsilon\,\mathrm{q\_3}(r)\,\%3\,\mathrm{d}(\theta)\cos(\theta)^2$$
$$+\,6\,r^3\,\%2\,\mathrm{d}(\theta)\,\varepsilon\,\omega(r)\cos(\theta)\,\%3\,du$$

$+ 6\, r^3 \sin(\theta)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du\, \varepsilon\, \mathrm{q\_3}(r)\, \%3\, \mathrm{d}(\theta) - 6\, r^3\, \mathrm{d}(\psi)\, \varepsilon\, \omega(r) \cos(\theta)\, \%3\, du$
$+ 6\, r^3\, \mathrm{d}(\psi)\, \varepsilon\, \omega(r) \cos(\theta)^3\, \%3\, du - 6\, r^3\, \%2\, \mathrm{d}(\theta)\, \varepsilon\, \omega(r) \cos(\theta)^3\, \%3\, du$
$+ 6\, r^3 \sin(\theta)\, \%2\, \mathrm{d}(\theta)^2\, \varepsilon\, \mathrm{q\_3}(r)\, \%3 - 6\, r^3\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du^2\, \varepsilon\, \omega(r) \cos(\theta)^3\, \%3$
$+ 6\, r^3\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du^2\, \varepsilon\, \omega(r) \cos(\theta)\, \%3$
$+ 6\, r^3 \sin(\theta)\, \mathrm{d}(\psi)\, \varepsilon\, \mathrm{q\_3}(r)\, \%3\, \mathrm{d}(\theta) \cos(\theta)^2 - 6\, r^3 \sin(\theta)\, \mathrm{d}(\psi)\, \varepsilon\, \mathrm{q\_3}(r)\, \%3\, \mathrm{d}(\theta)$
$- 6\, r^3 \sin(\theta)\, \%2\, \mathrm{d}(\theta)^2\, \varepsilon\, \mathrm{q\_3}(r)\, \%3 \cos(\theta)^2 + 2\, r^3\, \mathrm{d}(\psi)\, \%2\, \mathrm{d}(\theta) \cos(\theta)^2$
$+ 2\, r^3\, \%2\, \mathrm{d}(\theta)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du - 2\, r^3\, \%2\, \mathrm{d}(\theta)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du \cos(\theta)^2$
$+ 2\, r^3\, \mathrm{d}(\psi)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du \cos(\theta)^2 + r^3\, \%2^2\, \mathrm{d}(\theta)^2 + r^3\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))^2\, du^2$
$- r^3\, \mathrm{d}(\psi)^2 \cos(\theta)^2 + 9\, r^3\, \varepsilon^2\, \mathrm{q\_3}(r)^2\, \%1\, \mathrm{d}(\theta)^2 - 18\, r^3\, \varepsilon^2\, \mathrm{q\_3}(r)^2\, \%1\, \mathrm{d}(\theta)^2 \cos(\theta)^2$
$- 9\, r^3\, \varepsilon^2\, \omega(r)^2 \cos(\theta)^4\, \%1\, du^2 + 18\, r^3 \sin(\theta)\, \varepsilon^2\, \omega(r) \cos(\theta)\, \%1\, du\, \mathrm{q\_3}(r)\, \mathrm{d}(\theta)$
$+ 9\, r^3\, \varepsilon^2\, \omega(r)^2 \cos(\theta)^2\, \%1\, du^2 - 18\, r^3 \sin(\theta)\, \varepsilon^2\, \omega(r) \cos(\theta)^3\, \%1\, du\, \mathrm{q\_3}(r)\, \mathrm{d}(\theta))/r$
$\%1 := e^{(2\, i\, \sigma\, (u + \mathrm{F}(r)))}$
$\%2 := \frac{\partial}{\partial \theta}\, \mathrm{g}(r,\, \theta,\, u)$
$\%3 := e^{(i\, \sigma\, (u + \mathrm{F}(r)))}$

```
>  a19:=subs(epsilon^2=0,a18);
```

$a19 := (-2\, dr\, du\, r - du^2\, r + 2\, du^2\, M + r^3\, \mathrm{d}(\theta)^2 - 2\, r^3\, \mathrm{d}(\psi)\, \%1\, \mathrm{d}(\theta)$
$- 2\, r^3\, \mathrm{d}(\psi)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du - r^3\, \%1^2\, \mathrm{d}(\theta)^2 \cos(\theta)^2$
$- r^3\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))^2\, du^2 \cos(\theta)^2 + r^3\, \mathrm{d}(\psi)^2$
$- 6\, r^3 \sin(\theta)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du\, \varepsilon\, \mathrm{q\_3}(r)\, \%2\, \mathrm{d}(\theta) \cos(\theta)^2$
$+ 6\, r^3\, \%1\, \mathrm{d}(\theta)\, \varepsilon\, \omega(r) \cos(\theta)\, \%2\, du$
$+ 6\, r^3 \sin(\theta)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du\, \varepsilon\, \mathrm{q\_3}(r)\, \%2\, \mathrm{d}(\theta) - 6\, r^3\, \mathrm{d}(\psi)\, \varepsilon\, \omega(r) \cos(\theta)\, \%2\, du$
$+ 6\, r^3\, \mathrm{d}(\psi)\, \varepsilon\, \omega(r) \cos(\theta)^3\, \%2\, du - 6\, r^3\, \%1\, \mathrm{d}(\theta)\, \varepsilon\, \omega(r) \cos(\theta)^3\, \%2\, du$
$+ 6\, r^3 \sin(\theta)\, \%1\, \mathrm{d}(\theta)^2\, \varepsilon\, \mathrm{q\_3}(r)\, \%2 - 6\, r^3\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du^2\, \varepsilon\, \omega(r) \cos(\theta)^3\, \%2$
$+ 6\, r^3\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du^2\, \varepsilon\, \omega(r) \cos(\theta)\, \%2$
$+ 6\, r^3 \sin(\theta)\, \mathrm{d}(\psi)\, \varepsilon\, \mathrm{q\_3}(r)\, \%2\, \mathrm{d}(\theta) \cos(\theta)^2 - 6\, r^3 \sin(\theta)\, \mathrm{d}(\psi)\, \varepsilon\, \mathrm{q\_3}(r)\, \%2\, \mathrm{d}(\theta)$
$- 6\, r^3 \sin(\theta)\, \%1\, \mathrm{d}(\theta)^2\, \varepsilon\, \mathrm{q\_3}(r)\, \%2 \cos(\theta)^2 + 2\, r^3\, \mathrm{d}(\psi)\, \%1\, \mathrm{d}(\theta) \cos(\theta)^2$
$+ 2\, r^3\, \%1\, \mathrm{d}(\theta)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du - 2\, r^3\, \%1\, \mathrm{d}(\theta)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du \cos(\theta)^2$
$+ 2\, r^3\, \mathrm{d}(\psi)\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))\, du \cos(\theta)^2 + r^3\, \%1^2\, \mathrm{d}(\theta)^2 + r^3\, (\frac{\partial}{\partial u}\, \mathrm{g}(r,\, \theta,\, u))^2\, du^2$
$- r^3\, \mathrm{d}(\psi)^2 \cos(\theta)^2)/r$
$\%1 := \frac{\partial}{\partial \theta}\, \mathrm{g}(r,\, \theta,\, u)$
$\%2 := e^{(i\, \sigma\, (u + \mathrm{F}(r)))}$

```
>  a20:=factor(subs(d(theta)^2=0,dr^2=0,du*dr=0,dr=0,d(theta)=0,d(psi)=0
>  ,a19));
```

$$a20 := -du^2(r - 2\,M + r^3\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))^2\cos(\theta)^2$$
$$+ 6\,r^3\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,\varepsilon\,\omega(r)\cos(\theta)^3\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$
$$- 6\,r^3\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,\varepsilon\,\omega(r)\cos(\theta)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} - r^3\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))^2)/r$$

```
> a21:=subs(du^2=0,dr^2=0,d(theta)=0,d(psi)=0,a19);
```

$$a21 := -2\,dr\,du$$

```
> a22:=factor(subs(du^2=0,dr=0,d(psi)=0,d(theta)^2=0,a19));
```

$$a22 := -2r^2\,du\,\mathrm{d}(\theta)\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)(3\,(\tfrac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u))\,\varepsilon\,\omega(r)\cos(\theta)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$
$$+ 3\sin(\theta)\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} + (\tfrac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u))\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u)))$$

```
> a23:=factor(subs(du^2=0,dr=0,d(psi)^2=0,d(theta)=0,a19));
```

$$a23 := 2r^2\,\mathrm{d}(\psi)\,du\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)$$
$$(3\,\varepsilon\,\omega(r)\cos(\theta)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} + (\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u)))$$

```
> a24:=factor(subs(du^2=0,du=0,dr=0,d(psi)^2=0,d(theta)^2=0,a19));
```

$$a24 := 2r^2\,\mathrm{d}(\psi)\,\mathrm{d}(\theta)\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)$$
$$(3\sin(\theta)\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} + (\tfrac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)))$$

```
> a25:=factor(subs(du^2=0,dr=0,du=0,d(psi)=0,a19));
```

$$a25 := -r^2\,\mathrm{d}(\theta)^2(-1 + \%1^2\cos(\theta)^2 - 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$
$$+ 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\cos(\theta)^2 - \%1^2)$$
$$\%1 := \tfrac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)$$

```
> a26:=factor(subs(du^2=0,dr=0,du=0,d(theta)=0,a19));
```

$$a26 := -r^2\,\mathrm{d}(\psi)^2\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)$$

```
> a27:=coeff(a25,d(theta)^2);
```

$$a27 := -r^2(-1 + \%1^2\cos(\theta)^2 - 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$
$$+ 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\cos(\theta)^2 - \%1^2)$$
$$\%1 := \tfrac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)$$

```
> a28:=coeff(a26,d(psi)^2);
```

$$a28 := -r^2\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)$$

```
> a29:=lcoeff(a24,[d(psi),d(theta)]);
```

$$a29 := 2\,r^2\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)\,(3\sin(\theta)\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} + (\tfrac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)))$$

```
> a30:=a27*a28-(a29)^2=R^4*sin(theta)^2;
```

$$a30 := r^4(-1 + \%1^2 \cos(\theta)^2 - 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$
$$+ 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\cos(\theta)^2 - \%1^2)(\cos(\theta)-1)(\cos(\theta)+1)$$
$$-4\,r^4(\cos(\theta)-1)^2(\cos(\theta)+1)^2(3\sin(\theta)\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} + \%1)^2 =$$
$$R^4\sin(\theta)^2$$
$$\%1 := \frac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)$$

```
> a31:=factor(a30);
```

$$a31 := -r^4(\cos(\theta)-1)(\cos(\theta)+1)(3\,\%2^2\cos(\theta)^2 + 36\cos(\theta)^2\sin(\theta)^2\,\varepsilon^2\,\mathrm{q\_3}(r)^2\,\%1^2$$
$$+ 18\sin(\theta)\,\%2\,\varepsilon\,\mathrm{q\_3}(r)\,\%1\cos(\theta)^2 + 1 - 18\sin(\theta)\,\%2\,\varepsilon\,\mathrm{q\_3}(r)\,\%1 - 3\,\%2^2$$
$$- 36\sin(\theta)^2\,\varepsilon^2\,\mathrm{q\_3}(r)^2\,\%1^2) = R^4\sin(\theta)^2$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$
$$\%2 := \frac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)$$

```
> a32:=subs(epsilon^2=0,a31);
```

$$a32 := -r^4(\cos(\theta)-1)(\cos(\theta)+1)(3\,\%1^2\cos(\theta)^2 + 1$$
$$+ 18\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\cos(\theta)^2$$
$$- 18\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} - 3\,\%1^2) = R^4\sin(\theta)^2$$
$$\%1 := \frac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)$$

```
> a33:=factor(a32);
```

$$a33 := -r^4(\cos(\theta)-1)(\cos(\theta)+1)(3\,\%1^2\cos(\theta)^2 + 1$$
$$+ 18\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\cos(\theta)^2$$
$$- 18\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} - 3\,\%1^2) = R^4\sin(\theta)^2$$
$$\%1 := \frac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)$$

```
> a34:=R=r*(3*diff(g(r,theta,u),theta)^2*cos(theta)^2+18*sin(theta)*dif
> f(g(r,theta,u),theta)*epsilon*q_3(r)*e^(i*sigma*(u+F(r)))*cos(theta)^2
> +1-18*sin(theta)*diff(g(r,theta,u),theta)*epsilon*q_3(r)*e^(i*sigma*(u
> +F(r)))-3*diff(g(r,theta,u),theta)^2)^(1/4);
```

$$a34 := R = r(3\,\%1^2\cos(\theta)^2 + 1 + 18\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\cos(\theta)^2$$
$$- 18\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))} - 3\,\%1^2)^{(1/4)}$$
$$\%1 := \frac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)$$

```
> a35:=factor(subs(a16,a34));
```

$$a35 := R = r(3\,\%2^2\cos(\theta)^2 + 1 + 18\sin(\theta)\,\%2\,\varepsilon\,\mathrm{q\_3}(r)\,\%1\cos(\theta)^2$$
$$-\,18\sin(\theta)\,\%2\,\varepsilon\,\mathrm{q\_3}(r)\,\%1 - 3\,\%2^2)^{(1/4)}$$
$$\%1 := e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$
$$\%2 := \frac{\partial}{\partial\theta}\left(-3\,\varepsilon\cos(\theta)\int\frac{\%1\,(-\omega(r)\,r - \mathrm{q\_2}(r)\,r + 2\,\mathrm{q\_2}(r)\,M)}{-r+2\,M}\,dr\right)$$

```
> a36:=(subs(epsilon^2=0,a35));
```

$$a36 := R = r$$

```
> a37:=simplify(a36,trig);
```

$$a37 := R = r$$

```
> a38:=dr=dR;
```

$$a38 := dr = dR$$

```
> a39:=a20+a21+a22+a23+a24+a25+a26;
```

$$a39 := -du^2(r - 2\,M + r^3\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))^2\cos(\theta)^2 + 6\,r^3\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,\varepsilon\,\omega(r)\cos(\theta)^3\,\%2$$
$$-\,6\,r^3\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,\varepsilon\,\omega(r)\cos(\theta)\,\%2 - r^3\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))^2)/r - 2\,dr\,du - 2\,r^2\,du$$
$$\mathrm{d}(\theta)\,(\cos(\theta)-1)\,(\cos(\theta)+1)$$
$$(3\,\%1\,\varepsilon\,\omega(r)\cos(\theta)\,\%2 + 3\sin(\theta)\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u))\,\varepsilon\,\mathrm{q\_3}(r)\,\%2 + \%1\,(\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u)))$$
$$+\,2\,r^2\,\mathrm{d}(\psi)\,du\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(3\,\varepsilon\,\omega(r)\cos(\theta)\,\%2 + (\tfrac{\partial}{\partial u}\,\mathrm{g}(r,\,\theta,\,u)))$$
$$+\,2\,r^2\,\mathrm{d}(\psi)\,\mathrm{d}(\theta)\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(3\sin(\theta)\,\varepsilon\,\mathrm{q\_3}(r)\,\%2 + \%1) - r^2\,\mathrm{d}(\theta)^2($$
$$-1 + \%1^2\cos(\theta)^2 - 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,\%2 + 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(r)\,\%2\cos(\theta)^2$$
$$-\,\%1^2) - r^2\,\mathrm{d}(\psi)^2\,(\cos(\theta)-1)\,(\cos(\theta)+1)$$
$$\%1 := \frac{\partial}{\partial\theta}\,\mathrm{g}(r,\,\theta,\,u)$$
$$\%2 := e^{(i\,\sigma\,(u+\mathrm{F}(r)))}$$

```
> a40 := R = r;
```

$$a40 := R = r$$

```
> a41:=r=R;
```

$$a41 := r = R$$

```
> a42:=subs(a38,a41,a39);
```

$$a42 := -du^2(R - 2\,M + R^3\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))^2\cos(\theta)^2$$
$$+ 6\,R^3\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\,\varepsilon\,\omega(R)\cos(\theta)^3\,\%2 - 6\,R^3\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\,\varepsilon\,\omega(R)\cos(\theta)\,\%2$$
$$- R^3\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))^2)/R - 2\,dR\,du - 2\,R^2\,du\,\mathrm{d}(\theta)\,(\cos(\theta)-1)\,(\cos(\theta)+1)($$
$$3\,\%1\,\varepsilon\,\omega(R)\cos(\theta)\,\%2 + 3\sin(\theta)\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\,\varepsilon\,\mathrm{q\_3}(R)\,\%2$$
$$+ \%1\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u)))$$
$$+ 2\,R^2\,\mathrm{d}(\psi)\,du\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(3\,\varepsilon\,\omega(R)\cos(\theta)\,\%2 + (\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u)))$$
$$+ 2\,R^2\,\mathrm{d}(\psi)\,\mathrm{d}(\theta)\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(3\sin(\theta)\,\varepsilon\,\mathrm{q\_3}(R)\,\%2 + \%1) - R^2\,\mathrm{d}(\theta)^2$$
$$(-1 + \%1^2\cos(\theta)^2 - 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(R)\,\%2 + 6\sin(\theta)\,\%1\,\varepsilon\,\mathrm{q\_3}(R)\,\%2\cos(\theta)^2$$
$$- \%1^2) - R^2\,\mathrm{d}(\psi)^2\,(\cos(\theta)-1)\,(\cos(\theta)+1)$$
$$\%1 := \frac{\partial}{\partial \theta}\,\mathrm{g}(R,\,\theta,\,u)$$
$$\%2 := e^{(i\,\sigma\,(u+\mathrm{F}(R)))}$$

```
> simplify(coeff(a42,epsilon,0));
```

$$-(du^2\,R - 2\,du^2\,M + du^2\,R^3\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))^2\cos(\theta)^2 - du^2\,R^3\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))^2 + 2\,dR\,du\,R$$
$$+ 2\,R^3\,du\,\mathrm{d}(\theta)\,\%1\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\cos(\theta)^2 - 2\,R^3\,du\,\mathrm{d}(\theta)\,\%1\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))$$
$$- 2\,R^3\,\mathrm{d}(\psi)\,du\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\cos(\theta)^2 + 2\,R^3\,\mathrm{d}(\psi)\,du\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))$$
$$- 2\,R^3\,\mathrm{d}(\psi)\,\mathrm{d}(\theta)\,\%1\cos(\theta)^2 + 2\,R^3\,\mathrm{d}(\psi)\,\mathrm{d}(\theta)\,\%1 - R^3\,\mathrm{d}(\theta)^2$$
$$+ R^3\,\mathrm{d}(\theta)^2\,\%1^2\cos(\theta)^2 - R^3\,\mathrm{d}(\theta)^2\,\%1^2 + R^3\,\mathrm{d}(\psi)^2\cos(\theta)^2 - R^3\,\mathrm{d}(\psi)^2)/R$$
$$\%1 := \frac{\partial}{\partial \theta}\,\mathrm{g}(R,\,\theta,\,u)$$

```
> simplify(coeff(a42,epsilon,1));
```

$$-6R^2\,e^{(i\,\sigma\,(u+\mathrm{F}(R)))}(du^2\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\,\omega(R)\cos(\theta)^3 - du^2\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\,\omega(R)\cos(\theta)$$
$$+ du\,\mathrm{d}(\theta)\cos(\theta)^3\,\%1\,\omega(R) + du\,\mathrm{d}(\theta)\cos(\theta)^2\sin(\theta)\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\,\mathrm{q\_3}(R)$$
$$- du\,\mathrm{d}(\theta)\,\%1\,\omega(R)\cos(\theta) - du\,\mathrm{d}(\theta)\sin(\theta)\,(\frac{\partial}{\partial u}\,\mathrm{g}(R,\,\theta,\,u))\,\mathrm{q\_3}(R)$$
$$- \mathrm{d}(\psi)\,du\,\omega(R)\cos(\theta)^3 + \mathrm{d}(\psi)\,du\,\omega(R)\cos(\theta) - \mathrm{d}(\psi)\,\mathrm{d}(\theta)\sin(\theta)\,\mathrm{q\_3}(R)\cos(\theta)^2$$
$$+ \mathrm{d}(\psi)\,\mathrm{d}(\theta)\sin(\theta)\,\mathrm{q\_3}(R) + \mathrm{d}(\theta)^2\sin(\theta)\,\%1\,\mathrm{q\_3}(R)\cos(\theta)^2$$
$$- \mathrm{d}(\theta)^2\sin(\theta)\,\%1\,\mathrm{q\_3}(R))$$
$$\%1 := \frac{\partial}{\partial \theta}\,\mathrm{g}(R,\,\theta,\,u)$$

```
> a43:=(subs(diff(a16,u),a23));
```

$$a43 := 2r^2\,\mathrm{d}(\psi)\,du\,(\cos(\theta)-1)\,(\cos(\theta)+1)\left(3\,\varepsilon\,\omega(r)\cos(\theta)\,e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\right.$$
$$\left. - 3\,\varepsilon\cos(\theta)\int\frac{e^{(i\,\sigma\,(u+\mathrm{F}(r)))}\,i\,\sigma\ln(e)\,(-\omega(r)\,r - \mathrm{q\_2}(r)\,r + 2\,\mathrm{q\_2}(r)\,M)}{-r+2\,M}\,dr\right)$$

Appendix E

Maple computer programs 5

E.1 Program: th.map

This program works out J as a series in y as $J6(y)$ where $y = r - 2$. It also works out $J6(y)$ as a polynomial J. See chapter 5 sec. 5.2, which includes the result of the computation.

```
Order:=10;
M:=-MM;
n:=2:
b0:=0;
#om:=I*omm;
L2:=-n*(n+1); # \eth\bar{\eth}
L4:=L2^2+2*L2; #\eth^2\bar{ \eth}^2
D1r:=4*b0+r^3*diff(Z(r),r,r)+4*r^2*diff(Z(r),r)
+(L2+2)*r*diff(J(r),r)=0;
D2r:=-2*b0+r^2*diff(Z(r),r)+2*r*Z(r)-2*(r+M)*diff(J(r),r)
-r*(r+2*M)*diff(J(r),r,r)+2*r^2*om*diff(J(r),r)
+2*r*om*J(r)=0;
tr:={r=y+2*MM};
with(PDEtools);
```

```
D1y:=simplify(dchange(tr,D1r,[y],params=MM));
D2y:=simplify(dchange(tr,D2r,[y],params=MM));
D3y:=diff(D2y,y);
D1y1:=simplify(subs(diff(Z(y),y,y)=Z2,D1y)):
D1y2:=simplify(subs(diff(Z(y),y)=Z1,D1y1)):
D1y3:=simplify(subs(Z(y)=Z0,D1y2)):
D2y1:=simplify(subs(diff(Z(y),y,y)=Z2,D2y)):
D2y2:=simplify(subs(diff(Z(y),y)=Z1,D2y1)):
D2y3:=simplify(subs(Z(y)=Z0,D2y2)):
D3y1:=simplify(subs(diff(Z(y),y,y)=Z2,D3y)):
D3y2:=simplify(subs(diff(Z(y),y)=Z1,D3y1)):
D3y3:=simplify(subs(Z(y)=Z0,D3y2)):
MM:=1;
st:=solve({D1y3,D2y3,D3y3},{Z0,Z1,Z2}):
assign(st):
de:=simplify(diff(Z0,y)-Z1)=0;
de1:=simplify(subs(J(y)=int(J1(y),y),de)):
de2:=simplify(subs(J1(y)=diff(1/(y+2*MM),y)*J2(y),de1)):
de3:=simplify(subs(J2(y)=int(J3(y),y),de2));
assign(dsolve(de3,J3(y),series)):
J3(y):=subs(_C1=0,_C2=1,J3(y));
J4(y):=int(J3(y),y):
J5(y):=series(diff(1/(y+2*MM),y)*J4(y),y):
J6(y):=simplify(int(J5(y),y));
```

E.2 Program: ti.map

A Maple program that works out J as a series in x as *jout* where $x = 1/r$. It also works out *jout* as a polynomial jj. See chapter 5 sec. 5.2, which includes the result of the computation.

```
Order:=15;
#M is NEGATIVE
M:=-MM;
n:=2:
#om:=I*omm;
L2:=-n*(n+1); # \eth\bar{\eth}
L4:=L2^2+2*L2; #\eth^2\bar{ \eth}^2
D1r:=4*b0+r^3*diff(Z(r),r,r)+4*r^2*diff(Z(r),r)
+(L2+2)*r*diff(J(r),r)=0;
D2r:=-2*b0+r^2*diff(Z(r),r)+2*r*Z(r)-2*(r+M)*diff(J(r),r)
-r*(r+2*M)*diff(J(r),r,r)+2*r^2*om*diff(J(r),r)
+2*r*om*J(r)=0;
tr:={r=1/x};
with(PDEtools);
D1x:=simplify(dchange(tr,D1r));
D2x:=simplify(dchange(tr,D2r));
D3x:=diff(D2x,x);
D1x1:=simplify(subs(diff(Z(x),x,x)=Z2,D1x)):
D1x2:=simplify(subs(diff(Z(x),x)=Z1,D1x1)):
D1x3:=simplify(subs(Z(x)=Z0,D1x2)):
D2x1:=simplify(subs(diff(Z(x),x,x)=Z2,D2x)):
D2x2:=simplify(subs(diff(Z(x),x)=Z1,D2x1)):
D2x3:=simplify(subs(Z(x)=Z0,D2x2)):
D3x1:=simplify(subs(diff(Z(x),x,x)=Z2,D3x)):
D3x2:=simplify(subs(diff(Z(x),x)=Z1,D3x1)):
D3x3:=simplify(subs(Z(x)=Z0,D3x2)):
st:=solve({D1x3,D2x3,D3x3},{Z0,Z1,Z2}):
assign(st):
de:=simplify(diff(Z0,x)-Z1)=0;
```

```
de1:=subs(diff(J(x),x$4)=diff(J2(x),x$2),de):
de2:=subs(diff(J(x),x$3)=diff(J2(x),x),de1):
de3:=subs(diff(J(x),x$2)=J2(x),de2):
MM:=1;
d4:=lhs(de3);
s1:=series(d4,x,8);
JJ:=series(C3*x+sum(j[i]*x^i,i=3..Order),x,Order-2);
s3:=simplify(series(subs(J2(x)=JJ,d4/x^4),x));
for k from 3 to (Order-3) do
t1:=coeff(s3,x,k-3):
j[k]:=solve(t1=0,j[k]):
od:
JJ:=simplify(JJ);
jout:=series(int(int(JJ,x),x),x);
```

Bibliography

[1] Anderson A. and Price R. H., Phys. Rev. **D 43**, 3147 (1990).

[2] Anderson N., Proc. R. Soc. Lond. **A 439**, 47 (1992).

[3] Alder R., Bazin M. and Schiffer M., *Introduction to general relativity*, McGraw-Hill Book Co, New York, (1975).

[4] Baber W. G. and Hasser H. R., Proc. Camb. Phil. Soc. **25**, 564 (1935).

[5] Bachelot A. and Motet-Bachelot A., Ann. Inst. Henri Pointcare, **59**, 3, (1993). 2, 2, 1, 1, 6.1.1

[6] Babic M., Szilagyi B., Hawke I. and Zlochower Y., Class. Quantun Grav., **22**, 5089 (2005).

[7] Bardeen J.M. and Press W.H., J. Math. Phys., **14**, 7 (1972).

[8] Bauer C., Frink A. and Kerckel R., J. Symbolic Computation **33**, 1 (2002).

[9] Bishop N. T., Gómez R., Lehner L. and Winicour J., Phys. Rev. **D 54**, 6153 (1996).

[10] Bishop N. T., Gómez R.,Lehner L., Maharaj M. and Winicour J., Phys. Rev. **D 56**, 6298 (1997).

[11] Bishop N. T., Gómez R., Lehner L., Maharaj M. and Winicour J., Phys. Rev. **D 60**, 024005 (1999).

[12] Bishop N. T., Beyer F. and Koppitz M., Phys. Rev. **D 69**, 064010 (2004).

[13] Bishop N.T, Class. Quantun Grav., **22**, 2393 (2005).

[14] Bishop N. T. and Venter L. R., Phys. Rev. **D 73**, 084023 (2006).

[15] Bondi H., *Nature*, **186**, 535 (1960).

[16] Bondi H., van der Burg M. J. G. and Metzner A. W. K., Proc. R. Soc. London **A269**, 21 (1962).

[17] Campanelli M., Gø'mez R., Husa S., Winicour J. and Zlochower Y., Phys. Rev. **D 63**, 124013 (2001).

[18] Campanelli M. and Lousto C. O., Phys. Rev. **D 59**, 124022 (1999).

[19] Carroll S. M., *Lecture notes on general relativity*, University of California, Santa Barbara, CA 93106, (1997).

[20] Chandrasekhar S. and Friedmann J. L., Astrophys. J. **175**, 379 (1972).

[21] Chandrasekhar S., Proc. R. Soc. London, **A**, **343**, 289 (1975).

[22] Chandrasekhar S. and Detweiler S., Proc. R. Soc. London, **A**, **344**, 441 (1975).

[23] Chadrasekhar S., Proc. R. Soc. London, **A**, **369**, 425 (1980).

[24] Chandrasekhar S., *The Mathematical Theory of Black Holes and colliding waves*, Oxford University Press, New York, (1983).

[25] Chandrasekhar S., *The Mathematical Theory of Black Holes*, Oxford University Press, New York, (1983).

[26] Cunningham C. T., Price R. H. and Moncrief V., Astrophys. J., **224**, 643 (1978).

[27] Davies M., Ruffini R., Pres W. H. and Price R. H., Phys. Rev. Lett., **27**, 1466 (1971).

[28] Decarreau A., Maroni P. and Robert A., *Heun's Differential Equations* (ed Roneaux A.), Oxford University Press, (1995).

[29] Detweiler S., In *Sources of gravitational radiation* (ed. Smarr L.), Cambridge University Press, (1979).

[30] d'Inverno R.A, "A review of Algebraic Computing in General Relativity" in *General Relativity and Gravitation*, Volume 1, ed. A. Held, Plenum, 491 (1980).

[31] *É*anna *É*. F. and Scott A. H., New J. Phys. **7**, 204 (2005).

[32] Echeverria F., Phys. Rev. **D 40**, 3194 (1989).

[33] Edelstein L.A. and Vishveshwara C.V., Phys. Rev., **1**, 3514 (1969).

[34] Ehlers J., *Survey of General Relativity Theory, in Relativity, Astrophysics and Cosmology*, ed. W. Israel, D. Reidel, Dordrecht (1973).

[35] Ehlers J., Prasanna A.R. and Breuer R.A., Class. Quantum Grav., **4**, 253 (1987).

[36] Fiziev P. P., Class. Quantum Grav., **23**, 2447 (2006).

[37] Foster J. and Nightingale J.D., *A short cource in general relativity*, Springer-Verlag, New York, Inc.,(1995).

[38] Friedman J. L., *Proc. R. Soc. Lond.*, **A 335**, 163 (1973).

[39] Frittelli S. and Newmann, E.T., Phys. Rev. **D 59**, 124001 (1999).

[40] Gautschi W., SIAM Rev. **9**, 24 (1967).

[41] Gómez R., Lehner L., Papadopoulos P. and Winicour J., Class. Quantum Grav. **14**, 977 (1997).

[42] Gómez R., Marsa. and Winicour J., Phys. Rev. **D 56**, 6310 (1997).

[43] Golberg J.N., MacFarlane A.J., Newman E.T., Rohrlich F. and Sudarshan E.C.G., J. Math. Phys. **8**, 2155 (1967).

[44] Hall G.S. and Pulham J.R., *General Relativity*, JW Arrowsmith Ltd, Bristol, (1996).

[45] Hawking S. W. and Ellis G. F. R., *Large scale structure of space-time*, Cambridge University Press, Cambridge, England, (1973).

[46] http://www.npac.syr.edu/projects/bh/.

[47] Hughston L.P. and Tod K.P., *An Introduction to General Relativity*, 1st ed., Cambridge University Press, Cambridge, England, (1990).

[48] Husa S., Zlochower Y., Gòmez R. and Winicour J., Phys. Rev. **D 65**, 084034 (2002).

[49] Isaacson R.A., Welling J.S and Winicour J., J. Math. Phys. **26** , 2859 (1985).

[50] Janis A.I. and Porter J.R.(Ed), *Recent Advances in General Relativity*, Newton Center, MA, **4** (1992).

[51] Kenyon I. R.,*General Relativity*, Oxford University Press,Oxford (1990).

[52] Kokkotas K. D. and Schmidt B. G., Living Rev. Relativity, **2**, 2 (1999).

[53] Kruskal M. D., Phys. Rev. **119**, 1743 (1960).

[54] Landau L. and Lifshitz E., *The classical theory of fields*, 4th. edition, Pergamon (1977).

[55] Leaver E. W., Pro. R. Soc. London **A402**, 285 (1985).

[56] Lehner L., PhD. thesis, University of Pittsburgh (1998).

[57] Lopez-Aleman R., AMS/IP Stud. Adv Math. **13**, 377 (1999).

[58] Lipschutz M. M, Schaum's outline of *Theory and problems of differential geometry*, McGraw-Hill, Inc., New York, (1969).

[59] Moncrief V., Ann. Phys. (N.Y.) **88**, 323 (1973).

[60] Misner C., Thorne K. S. and Wheeler J. A., *Gravitation*, Freeman, San Francisco, (1973).

[61] Narlika J. V., *Introduction to cosmology*, 2nd. edition, Cambridge University Press, Cambridge, (1993)

[62] Newman E. T. and Perose R., J. Maths. Phys. **3**, 566 (1962).

[63] Newman E. T. and Penrose R., J. Math. Phys. **7**, 863 (1966).

[64] ÓNeill B., *Elementary differential geometry*, 2^{nd}, Academic Press, San Diego, CA, (1997).

[65] Penrose R., Phys. Rev. Lett. **10**, 66 (1963).

[66] Penrose R., *Structure of spacetime*, In *Battelle rencontres* (ed. DeWitt C. M. and Wheeler J. A.), Benjamin, New York (1968).

[67] Poisson E., Phys. Rev. **D 55**, 639 (1997).

[68] Price R. H., Phys. Rev. **D 5**, 2419 (1972).

[69] http:reduce-algebra.com

[70] Regge T. and Wheeler J., Phys. Rev. **108**, 1063 (1957).

[71] Gleiser J.R., Nicasio C.O., Price R.H. and Pullin J., Phys. Rept., **325**, 41 (2000).

[72] Rutter J. W, *Geometry of curves*, Chapman and Hall/CRC Press, Boca Raton, Florida, (200).

[73] Sachs R. K., Proc. R. Soc. London **A 270**, 103 (1962).

[74] Sachs R. K., Phys. Rev. Lett. **150**, 66 (1963).

[75] Sammett J. E, *Commum. ACM*, **9**, 555 (1966).

[76] Sasaki M. and Nakamura T., Phys. Lett. A, **89**, 68 (1982).

[77] Sasaki M. and Nakamura T., Prog. Theor. Phys., **67**, 1788 (1982).

[78] Schmidt B.G.(Ed.), *Einstein's Field Equations and Their Physical Implications*, Golm, Germany, (2000).

[79] Schmidt B.G., Zeits. f. Naturfor, **22a**, 1351 (1967).

[80] Schutz B.F., *A first cource in general relativity*, Cambridge University Press, Cambridge, (1985).

[81] Schwarzchild K., *Ü*ber das Gravitationsfield eines Masspunktes nach der Einsteinschen Theorie. *Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math-Phys. Tech.*, 189 (1916).

[82] Seidel E., appeared in *"On the Black Hole Trail"* eds. Bala Iyer and Biplab Bhawal (Kluwer), Report-no: AEI-066 (1998).

[83] Seidel E., plenary talk given at GR15, Poona, India, appeared in the proceedings, Report-no: AEI-071 (1998).

[84] Sopuerta C. F. and Laguna P., Phys. Rev. **D 73**, 044028 (2006).

[85] Teukolsky S.A., Astrophys. J., **185**, 635 (1973).

[86] Tamburino A. and Winicour J. Phys. Rev. **150**, 1039 (1966).

[87] Tominaga K., Saijo M. and Maeda K., Phys. Rev. **D 63**, 124012 (2001).

[88] Townsend P.K., *Lecture notes on black holes*, Univ. of Cambridge, Silver St., Cambridge, U.K. (1997).

[89] Vishveshwara C. V., Phys. Rev. **D 1**, 2870 (1970).

[90] Vishveshwara C. V., Nature **227**, 936 (1970).

[91] Wald R.M.,*General relativity*, University of Chicago Press, Chicago, (1984).

[92] Wald R.M., Phys. Rev. Lett., **41**, 203 (1978).

[93] Weinberg S., *Gravitation and cosmology: Principles and applications of the general theory of relativity*, Wiley and Sons, New York, (1972).

[94] Wheeler J., Phys. Rev. **97**, 511 (1955).

[95] Wilson A. H., Proc, R. Soc. **A**, **118**, 617 (1928).

[96] Winicour J., J. Math. Phys. **24**, 1193 (1983).

[97] Winicour J., J. Math. Phys. **25**, 2506 (1984).

[98] Winicour J., Living Reviews (2001).

[99] Zerilli F.J., Phys. Rev. **D 2**, 2141 (1970).

[100] Zerilli F. J., Phys. Rev. Lett. **24**, 737 (1970).

[101] Zlochower R., Gomez R., Husa l., Lehner L. and Winicour J., Phys. Rev. **D 68**, 084014 (2003).